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LINEAR PROGRAMMING*

BY

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International Business Machines Corporation, New York

1. Economic foundation. The growing application of mathematical techniques to problems of economics has brought about, in the last decade, development of a practical and promising method for providing answers to certain, hitherto unanswerable questions. Where applicable, "Linear Programming" is able to divulge the most profitable or least costly mode of operations (within given ranges of conditions) for a variety of industrial operations. At times, unexpected results obtained thereby represented an improvement over past operations and permitted substantial savings thereafter. The mathematical formulation and solution of problems expressible in Linear Programming form promises to grow into a useful servant to the economic planning of many an industry.

Among the questions to which Linear Programming can help provide answers, are these:

1. Within specified ranges of conditions, what quantities of what items should be manufactured so as to yield maximum profit?

2. Within stipulated ranges of specifications, what mixture of what compositions should be marketed so as to result in maximum profit? Or in maximum sales volume?

3. Within definite ranges of market conditions, from what suppliers shall what items be distributed to what consumers so as to minimize costs of distribution?

4. Within specified limitations of labor requirements, what allocation of what labor categories to what jobs will require a minimum of operating personnel? Or minimum expenditure? Or assure maximum efficiency?

5. Given the nutritional values of various animal feeds, what quantities of what feeds should be selected so as to satisfy minimum nutritional requirements and minimize overall cost of raising livestock?

With complicated interdependent industrial operations, a change in one operating characteristic may propagate as a network of readjustments in other associated factors, the tracing of which may become excessively intricate. Trial and error methods then prove impractical and a systematic procedure is called for.

The restrictions or constraints imposed may be of a most diverse nature and may include the following: raw material availability; plant capacity; transportation facilities; freight rates for ships, railroads, trucks, and pipelines; lengths of shipping routes; storage capacities and costs; product disposability; time limitations on supply, processing, storage, and shipments; balance between input and output; physical, chemical, or tech-

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nological quality specifications such as volatilities, vapor pressure, viscosities, assay, quality performance, etc.; and others.

In many cases such restrictions take the mathematical form of inequalities with non-negative variables. The problem consists in determining, out of the infinite number of possible solutions, that unique solution which makes a certain function (profit, or volume, or quality, or efficiency, or costs, or other requirement) a maximum; or a minimum. The results will then provide the best possible planning of operations under the specified restrictions. Whenever the inequalities encountered are *linear* in the variables, the problem is said to be one of "linear programming" and it is to this type of problem that the present paper applies.

Once the problem has been formulated, the actual solution consists of a simple arithmetical routine procedure which can be learned by non-technical personnel and requires no knowledge of mathematics whatever. While the practical procedure for solution is easy to learn and apply, the following pages present a detailed mathematical derivation.*

2. Mathematical formulation. The mathematical formulation of the typical problem of linear programming is as follows.

Find values $x_1 , x_2 , \cdots x_n$ which satisfy the conditions

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &\leqq b_1 , \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &\leqq b_2 , \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &\leqq b_m , \end{aligned} \tag{2.1}$$

$$x_i \geqq 0 \qquad \text{for all } i \tag{2.2}$$

and such as to make

$$f = c_1x_1 + c_2x_2 + \cdots + c_nx_n = \text{a maximum}, \tag{2.3}$$

where a_{ij} , b_i , c_i are given constants.

Denoting scalars by small-case letters, column vectors by capitals, and defining A_j as the jth column of the coefficients matrix A, (1) may be restated in matrix form,

$$A_1x_1 + \cdots + A_nx_n \leqq B. \tag{2.4}$$

If a problem prescribes *minimization* of a function instead of maximization, the requirement $\phi = \sum d_ix_i = \text{minimum}$ can be converted into the maximization requirement (3) by the substitution $d_i = -c_i$,

$$f = -\phi = \sum c_ix_i = \text{maximum}. \tag{2.5}$$

Equation (1) may also include equations $\sum a_{ij}x_j = b_i$ or inequalities of the form $\sum a_{ij}\, x_j \geqq b_i$.

More general problems which do not require restriction (2) for some or all of the

*Equations quoted by single numbers [e.g. (4)] refer to the same section as the discussing text. References to equations in other sections are indicated by twin numbers [e.g. (7.5)].

variables, can always be reduced to the present form by the substitution $x_i = y_i - z_i$ with $y_i, z_i \geqq 0$ so that restriction (2) now applies to the new variables.

Any inequality of the form $x_i \geqq b_i$ may be converted, by the substitution $x_i = b_i + y_i$ into $y_i \geqq 0$, which is of the form (2). It will thus be respected automatically by the subsequent procedure and need no longer be carried explicitly under (1).

Illustrative example. We shall illustrate our discussion by carrying along and solving one numerical example, which is tiny in magnitude but which illustrates the procedure to be followed for large-scale problems.

Suppose for extreme simplicity that a manufacturer produces four items, x_1 to x_4. Let the economic limitations take the form,

$$\begin{aligned} 2x_3 + x_4 &= 10, \\ -7x_1 + x_2 + 2x_3 - 5x_4 &= 14, \\ 6x_1 - 4x_3 + x_4 &\leqq 4. \end{aligned} \tag{2.6}$$

Non-negative production requirements give the conditions,

$$\text{all } x_i \geqq 0. \tag{2.7}$$

Let the profit for each manufactured item be as follows:

Item	x_1	x_2	x_3	x_4
Profit (in thousands of dollars)	4	-5 (= loss)	8	38

Of all possible production schedules meeting these conditions it is desired to find that one which will maximize profit, i.e. for which

$$f = 4x_1 - 5x_2 + 8x_3 + 38x_4 = \text{a maximum.} \tag{2.8}$$

Note that the values c_i are here interpreted as profit coefficients.

3. Conversion to standard equalities. The given inequalities (2.1) are first converted into equalities with non-negative right-hand sides as follows.

Rule 1. *Multiply through by* -1 *where necessary to make all* $b_i \geqq 0$.

Rule 2. *Where* $b_i = 0$, *multiply through by* -1 *where necessary to convert inequalities* $\sum a_{ij} x_j \geqq 0$ *into the form* $\sum a'_{ij} x_j \leqq 0$.

Rule 3. *Replace inequalities*

$$a_{i1}x_1 + \cdots + a_{in}x_n \leqq b_i \tag{3.1}$$

by the equalities

$$a_{i1}x_1 + \cdots + a_{in}x_n + x_{n+i} = b_i$$

and replace inequalities

$$a_{i1}x_1 + \cdots + a_{in}x_n \geqq b_i \tag{3.2}$$

by the equalities

$$a_{i1}x_1 + \cdots + a_{in}x_n - x_{n+i} = b_i$$

The additional variables x_{n+i} take up the "slack" by which the inequalities are allowed to depart from equalities and are therefore called "slack variables". From (1) and (2) they are subject, like all other variables x_i, to condition (2.2) and may constitute an integral part of the final solution. By (2.3) they will be able to make no positive contribution to f and must therefore be adjoined to the right hand side of (2.3) with associated coefficients $c_{n+i} = 0$. No slack variable is introduced for any *equality* of (2.1).

With the inclusion of original as well as slack variables, our problem now reads:

$$A_1x_1 + \cdots + A_{n'}x_{n'} = B, \qquad \text{where all} \qquad b_i \geqq 0, \tag{3.3}$$

$$x_i \geqq 0 \qquad \text{for all } i, \tag{3.4}$$

$$f = c_1x_1 + \cdots + c_{n'}x_{n'} = \max. \tag{3.5}$$

4. Solvability theorems. We are confronted with the simple problem of maximizing a linear function of several variables subject to side conditions, apparently amenable to the standard analytical treatment with Lagrange multipliers. However, an analytical maximum may well have some or all of the variables negative, violating restrictions (3.4). If we should try to satisfy (3.4) by the substitution $x_i = y_i^2$ and then apply Lagrange multipliers, the derivatives of (3.5) will turn out to be linear, but the constraint relations (3.3), which will specifically be needed for solution, will remain non-linear, furnishing a system of simultaneous non-linear equations. A different approach will therefore be taken.

We define:

A "feasible" solution is a set of x_i which satisfies (3.3) and (3.4) but not necessarily (3.5).

A "maximal" solution satisfies (3.3), (3.4) and (3.5).

A "basic feasible" solution is a feasible solution with *not more than* m non-zero (and therefore positive) values x_i .

A "basic maximal" solution is a maximal solution with not more than m non-zero (and therefore positive) values x_i .

The following existence theorems apply to our problem:

Theorem 1. Whenever there exists a feasible solution, there also exists a basic feasible solution.

Theorem 2. If f remains finite for all possible feasible solutions, then there exists a basic maximal solution.

Solution of the linear programming problem is accomplished by the so-called "simplex" procedure. First a basic feasible solution is constructed. This is then replaced by another basic feasible solution for which f is larger than before. The process is continued until either

(i) f can no longer increase, indicating attainment of a maximal solution; or

(ii) certain characteristics in the procedure indicate that $f_{\max} = \infty$.

The number of iterations required will be shown to be finite.

For the sake of illustration, we shall for the moment assume that our procedure admits of a solution and that $f_{\max}$ = finite. Other situations will be discussed further below.

5. Unit basis. If the first m column vectors A_j are linearly independent, they provide a basis in terms of which all columns of the coefficient matrix can be expressed as

$$A_j = d_{1j}A_1 + d_{2j}A_2 + \cdots + d_{mj}A_m \, . \tag{5.1}$$

If we can furthermore arrange it that $A_1, A_2, \cdots A_m$ are the m columns I_i of the mth order identity matrix (we shall call these vectors "unit columns", collectively forming a "unit basis"), we shall gain two vital advantages.

First, to obtain the d_{ij} , which will be required in (7.1), (8.1) and (8.2) below, it would normally be necessary to solve Eq. (1), which, written in terms of the components

$a_{1i} \cdots a_{mi}$ of the A_i, represents m equations in the m unknowns d_{ij}. Solutions will be required for $j = m + 1$ to n and it would thus be necessary to solve $n - m$ systems of m equations each. With $A_i = I_i$, this labor will be completely eliminated, for now

$$A_j = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \cdot \\ \cdot \\ \cdot \\ a_{mj} \end{bmatrix} = a_{1j} \begin{bmatrix} 1 \\ 0 \\ 0 \\ \cdot \\ \cdot \\ 0 \end{bmatrix} + a_{2j} \begin{bmatrix} 0 \\ 1 \\ 0 \\ \cdot \\ \cdot \\ 0 \end{bmatrix} + \cdots + a_{mj} \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ 0 \\ 1 \end{bmatrix} \qquad [\text{any } j], \tag{5.2}$$

so that $d_{ij} = a_{ij}$, which are given.

$$\left.\begin{array}{lll} \text{Second, with} & x_i = 0 \quad \text{for} & i > m, \\ \text{(3.3) then gives} & x_i = b_i \geqq 0 & (i = 1 \text{ to } m), \end{array}\right\} \tag{5.3}$$

immediately furnishing a basic feasible solution.

6. Initial set-up. We first wish to set up a starting unit basis. In (3.3) several of the originally given A_j may happen to be unit columns. Furthermore, each slack variable x_{n+i} supplied by (3.1) [but not by (3.2)] introduces a corresponding unit column A_{n+i}.

We now apply

Rule 4. *Augment the coefficient matrix by further unit columns I_i until A contains all the m possible unit columns of an $m \times m$ identity submatrix.*

Equation (3.3) is thus replaced by

$$A_1 x_1 + \cdots + A_{n'} x_{n'} + A_{n'+1} x_{n'+1} + \cdots + A_{n''} x_{n''} = B, \tag{6.1}$$

where the new unit columns and quantities $x_{n'+i}$ are called "artificial" vectors and variables.

Our example, by the introduction of slack and artificial variables, now takes the matrix form

$$\begin{bmatrix} 0 & 0 & 2 & 1 & 0 & 1 \\ -7 & 1 & 2 & -5 & 0 & 0 \\ 6 & 0 & -4 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 10 \\ 14 \\ 4 \end{bmatrix}. \tag{6.2}$$

Here

A_5 = slack vector $\qquad$ A_6 = artificial vector

x_5 = slack variable $\qquad$ x_6 = artificial variable

All 3 columns of the 3rd-order identity matrix now appear in A. Any inequality of the type (3.2) introduces a slack vector $-I_i$ into the matrix of coefficients and therefore will in addition require the introduction of an artificial vector $+I_i$.

Our original equations (2.1) permit certain amounts of slack, which may well appear in the final solution. Since they do not make any contribution to the function f, the c_i coefficients corresponding to *slack* variables x_i were taken as zero.

The *artificial* variables, however, are introduced only for convenience in getting started and have no place in a final solution. To assure their subsequent elimination we attach an unduly large penalty to their presence. By setting the c_i coefficients corresponding to *artificial* variables x_i equal to $-M$ (a very large negative number), we shall insure that f cannot possibly reach its maximum as long as any feasible solution still contains any of the artificial variables. Their elimination by the routine procedure below is thus insured automatically.

It is convenient to apply

Rule 5. Re-arrange the A-columns and variables x_i so as to display the identity submatrix on the left.

Redefining accordingly A_i , x_i , c_i , and n, our problem takes on the final formulation,

$$\left.\begin{aligned} A_1x_1 + A_2x_2 + \cdots + A_nx_n &= B \\ (A_j = I_j \quad \text{for} \quad j \leqq m; \quad \text{all} \quad b_i \geqq 0) &\end{aligned}\right\}, \tag{6.3}$$

$$\text{all} \quad x_i \geqq 0, \tag{6.4}$$

$$f = c_1x_1 + c_2x_2 + \cdots + c_nx_n = \max. \tag{6.5}$$

By (5.3), an initial basic feasible solution is immediately obtained from (6.3) as

$$x_i \begin{cases} = b_i & \text{for} \quad i \leqq m \\ = 0 & \text{for} \quad i > m, \end{cases} \tag{6.6}$$

which satisfies

$$A_1x_1 + \cdots + A_mx_m = B \qquad [A_j = I_j \text{ for all } j], \tag{6.7}$$

and

$$f = c_1x_1 + \cdots + c_mx_m \qquad \text{(not yet necessarily a maximum)}. \tag{6.8}$$

Our example, on defining

$$\begin{aligned} x_1' = x_6 \qquad x_2' = x_2 \qquad x_3' = x_5 \\ x_4' = x_4 \qquad x_5' = x_1 \qquad x_6' = x_3 \end{aligned} \tag{6.9}$$

now takes on the final form

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & -5 & -7 & 2 \\ 0 & 0 & 1 & 1 & 6 & -4 \end{bmatrix} \begin{bmatrix} x_1' \\ x_2' \\ \cdot \\ \cdot \\ \cdot \\ x_6' \end{bmatrix} = \begin{bmatrix} 10 \\ 14 \\ 4 \end{bmatrix} \tag{6.10}$$

$$\text{all} \quad x_i' \geqq 0 \tag{6.11}$$

$$f = -Mx_1' - 5x_2' + 0x_3' + 38x_4' + 4x_5' + 8x_6' = \max, \tag{6.12}$$

with the initial basic feasible solution

$$x_1' = 10, \qquad x_2' = 14, \qquad x_3' = 4, \qquad x_4' = x_5' = x_6' = 0. \tag{6.13}$$

7. Change of basis. In conformity with *Existence Theorem* 1, solution (6.6) involves non-zero variables x_i at most for $i = 1$ to m. We shall now try to determine x_r $(r \leqq m)$ and some value $x_k \geqq 0$ $(k > m)$ such that, in (6.3), replacement of the term A_rx_r by a term A_kx_k (A_k = given) will leave all previous conditions satisfied and also *increase* f. This new solution, again involving only m variables x_i, will thus also be "basic feasible", but the increase in f will represent an improvement.

Instead of concentrating on the variables x_i, we can express our intentions in terms of the given vectors A_i: desired to replace A_r $(r \leqq m)$, one of the unit base vectors, by A_k $(k > m)$, one of the matrix columns not in the basis, and to determine corresponding r and k, such that (6.3) and (6.4) will remain satisfied and (6.5) will increase. This process is called "introducing vector A_k into the basis".

From (5.2), for $j = k$,

$$A_k - (A_1a_{1k} \cdots + A_ma_{mk}) = 0. \tag{7.1}$$

In order to introduce A_k into the basis, we add θ times identity (1) to (6.7):

$$A_k\theta + A_1(x_1 - \theta a_{1k}) + \cdots + A_r(x_r - \theta a_{rk}) + \cdots + A_m(x_m - \theta a_{mk}) = B \tag{7.2}$$

Lemma 1. For $i = k$, restriction (6.4) applied to (2) requires $\theta \geqq 0$.

How will this modification affect f? From (6.8),

$$f = c_1x_1 + \cdots + c_mx_m\ . \tag{7.3}$$

In order to construct the new f corresponding to (2), we analogously define

$$f_j = c_1a_{1j} + \cdots + c_ma_{mj} \qquad [j = 1 \text{ to } n]. \tag{7.4}$$

Putting herein $j = k$, subtracting θ times (4) from (3), and adding the term $c_k\theta$ to each side, gives

$$f' \equiv f - \theta(f_k - c_k) = c_k\theta + c_1(x_1 - \theta a_{1k}) + \cdots + c_m(x_m - \theta a_{mk}). \tag{7.5}$$

The right-hand side is precisely the new f for (2); and we derive

Lemma 2. For $\theta > 0$ we can make $f' > f$ only if k is so chosen that $f_k - c_k < 0$.

Lemma 3. When eventually no negative $f_k - c_k$ remains, no further increase in f is possible. It will indicate termination of our procedure owing to attainment of a *maximal* solution.

Choice of k. In view of Lemma 2, it is convenient to replace (4) by the definition

$$g_j \equiv f_j - c_j = c_1a_{1j} + \cdots + c_ma_{mj} - c_j \qquad [j = 1 \text{ to } n]. \tag{7.6}$$

In theory, the largest increase in f, for one step, is obtained in (5) when $-\theta\,(f_k - c_k)$ is as large as possible. In practice we do the following, which is almost equally satisfactory and requires far less labor.

Rule 6. *Examine only the negative* $g_j \equiv f_j - c_j$, *choose the most negative one, and take that* j *as* k.

Choice of r. For all $i \leqq m$, requirement (6.4) applied to the coefficients of A_i in (2) becomes $x_i - \theta\, a_{ik} \geqq 0$.

Lemma 4. For $a_{ik} \leqq 0$, this requirement is automatically satisfied for any $\theta \geqq 0$, since $x_i \geqq 0$.

Lemma 5. For $a_{ik} > 0$, we must make $\theta \leqq x_i/a_{ik}$ for all $i \leqq m$.

Lemma 6. Elimination of A_r requires $x_r - \theta\, a_{rk} = 0$, i.e. for some r, $\theta = x_r/a_{rk}$.

Lemmas (1), (5), (6) will be simultaneously satisfied by

Rule 7. *For the chosen k, select only those a_{ik} which are > 0 to form the quotients $q_i = x_i/a_{ik}$ $(\geqq 0)$. Take r = the i corresponding to the smallest q_i.*

Hence

$$\theta = \frac{x_r}{a_{rk}} = \min_{a_{ik}>0} \frac{x_i}{a_{ik}}. \tag{7.7}$$

Example. It proves efficient to arrange numerical problems in the following table form, which here represents (6.10) to (6.12).

TABLE 1.

c_i of basis	Basis	c_j:	0	$-M$	-5	0	38	4	8
		Vector:	$B = A_0$	A_1	A_2	A_3	A_4	A_5	A_6
$-M$	A_1		10	1	0	0	1	0	2
-5	A_2		14	0	1	0	-5	-7	2
0	A_3		4	0	0	1	1	6	-4
$g_j = (\sum_i c_i a_{ij}) - c_j$:			$-10M$ -70	0	0	0	$-M$ -13	31	$-2M$ -18

Since $b_i = x_i = a_{i0}$, the insertion $c_0 = 0$ gives $g_0 = \sum c_i a_{i0} - 0 = \sum c_i x_i = f$ so that our table also displays f. Rule 6 gives $k = 6$; rule 7 gives $r = 1$; (7) gives $\theta = 5$.

In rule 6, $j = 0$ is excluded from consideration; in rule 7, $i = m + 1$ is excluded.

8. Transformation to new basis. Our change of the basis $A_1 \cdots A_r \cdots A_m$ consisted in "introducing A_k into the basis" and removing A_r. Each A_j, expressed in terms of the old basis by (5.1), must now be converted into a new form A_j' which similarly refers to the new basis.

By (5.2), for *any* j,

$$A_j = a_{1j}A_1 + \cdots + a_{rj}A_r + \cdots + a_{mj}A_m\,. \tag{8.1}$$

Putting $j = k$ and solving for A_r, we obtain $A_r = (1/a_{rk})\,(A_k - \sum_{i \neq r} a_{ik}A_i)$. Substitution into (1) will express *any* A_j in terms of the new basis, viz,

$$A_j' = \frac{a_{rj}}{a_{rk}}\,A_k + \sum_{i \neq r}\left(a_{ij} - \frac{a_{rj}}{a_{rk}}\,a_{ik}\right)A_i \qquad [\text{any } j]. \tag{8.2}$$

This transformation likewise applies to the vector $B \equiv A_0$. For $j = k$, (2) gives

$$A_k' = 1 \cdot A_k + \sum 0 \cdot A_i \tag{8.3}$$

as should be.

Restoration to unit form. The new basis now contains $A_k = [a_{1k}, \cdots, a_{mk}]$ and is therefore no longer a unit basis. However, a notational trick restores unit form as follows:

In (5.2), the rth component of any vector indicated the coefficient of A_r, which

henceforth will be zero. On the other hand we must adjoin to (5.2) an additional term $a'_{kj}A'_k$. We now utilize the available vacancy of the rth component a'_{rj} of each vector to indicate $a'_{kj} = a_{rj}/a_{rk}$, the coefficient of A_k [from (2)]. In this new notation, (3) becomes $A'_k = [0, \cdots, 0, 1, 0, \cdots, 0] = I_r$ where the component 1 occurs in row r and which is now *in unit form.*

For the other unit vectors A_i remaining in the basis ($j \leqq m, j \neq r$), we have

$$a_{ij}\begin{cases} = 0 & \text{when} \quad i \neq j \\ = 1 & \text{when} \quad i = j \end{cases}$$

and their new form is given by (2) as $A'_j = 0 + \sum_i (a_{ij} - 0)\, A_i = A_j = I_j$ [$j \leqq m$, $j \neq r$]. The transformation thus leaves them unchanged in their original unit form.

However, the former unit base vector $A_r (j = r)$ is turned by (2) into $A'_r = (1/a_{rk})\, A_k + \sum_{i \neq r} (-\alpha_{ik}/a_{rk})\, A_i$ and is thus no longer in unit form.

With our notation, (2) becomes, analogously to (5.2),

$$A'_j = \begin{bmatrix} a'_{1j} \\ \cdot \\ \cdot \\ \cdot \\ a'_{mj} \end{bmatrix} = \frac{a_{rj}}{a_{rk}} I_r + \sum_{i \neq r} \left(a_{ij} - \frac{a_{rj}}{a_{rk}} a_{ik} \right) I_i$$

and therefore, as in (5.2),

$$a'_{ij}\begin{cases} = a_{rj}/a_{rk} & \text{for} \quad i = r \\ = a_{ij} - (a_{rj}/a_{rk}) a_{ik} & \text{for} \quad i \neq r \end{cases} \qquad [\text{any } j], \tag{8.4}$$

which gives the law of transformation of individual components.

Transformation of g_j and f. Equation (5.2) gives rise to the definition (7.6),

$$g_j = \sum_{i=1}^{m} c_i a_{ij} - c_j \, .$$

Similarly from (2),

$$g'_j = c_k a'_{rj} + \sum_{i \neq r} c_i a'_{ij} - c_j \tag{8.5}$$

which, by (4), becomes

$$g'_j = c_k \frac{a_{rj}}{a_{rk}} + \sum_{i \neq r} c_i \left(a_{ij} - \frac{a_{rj}}{a_{rk}} a_{ik} \right) - c_j \, .$$

Adding the identity

$$0 \equiv c_r a_{rj} - (a_{rj}/a_{rk}) \cdot c_r a_{rk} \, ,$$

$$g'_j = \Big(\sum_{\text{all } i} c_i a_{ij} - c_j \Big) - \frac{a_{rj}}{a_{rk}} \Big(\sum_{\text{all } i} c_i a_{ik} - c_k \Big),$$

i.e.,

$$g'_j = g_j - \frac{a_{rj}}{a_{rk}} g_k \, . \tag{8.6}$$

For $j = 0$, $b_i = a_{i0}$ and $c_0 = 0$, (6) also gives f'.

From (7.6), $g_j = 0$ for $j = 1$ to m.

From (6), for the new basis vector k, likewise $g'_k = 0$ but for the vector r removed from the basis, the new value $g'_r = 0 - (1/a_{rk})\, g_k$ which (since $g_k < 0$, $a_{rk} > 0$) is > 0.

The identity of this transformation relation (6) with (4) (case $i \neq r$) allows us to establish

Rule 8. *Treating the table of $B \equiv A_0$, the A_j, and the row of g_j (including $j = 0$) all as one single matrix, apply transformation* (8.4) *as follows. Divide through row r by the pivotal element a_{rk}, giving the new row r. Replace each remaining element a_{ij} of the matrix, including the g_j row and including the column $j = 0$, by $a'_{ij} = a_{ij} - a'_{rj}\, a_{ik}$, which involves precisely the four corner points of a rectangle built on the diagonal line from a_{ij} to the pivotal element. A'_k then becomes I_r, g'_k becomes 0, basis vectors other than A_r remain unaltered. In the "basis identification" columns to the left of the matrix, replace c_r by c_k and A_r by A_k.*

After c_r in the left-hand margin has been replaced by c_k, an arithmetic check can be obtained by comparing the g'_j with those computed independently from (5), which now becomes

$$g'_j = \sum_{\text{all } i} c'_i a'_{ij} - c_j \,. \tag{8.7}$$

Example. One iteration of Table 1 by means of Rule 8 and check (7) results in

TABLE 2.

c_i of basis	Basis	c_j:	0	$-M$	-5	0	38	4	8
		Vector:	$B \equiv A_0$	A_1	A_2	A_3	A_4	A_5	A_6
8	A_6		5	1/2	0	0	1/2	0	1
-5	A_2		4	-1	1	0	-6	-7	0
0	A_3		24	2	0	1	3	6	0
$g_j = (\sum_i c_i a_{ij}) - c_j$:			20	$M + 9$	0	0	-4	31	0

$f = g_0$ is shown to be 20, A_0 and the identifying "basis" column display the improved basic feasible solution

$$x'_6 = 5, \qquad x'_2 = 4, \qquad x'_3 = 24, \qquad x'_1 = x'_4 = x'_5 = 0$$

and the presence of a negative g_j indicates, by Lemma 2, that further improvement is possible.

Subsequent iterations. After completing any iteration, there is no need to rearrange columns, as the above theory still holds for the columns as now standing, by identification of the I_i with the appropriate A_j. Further iterations are carried out by application of rules (6), (7), (8) to the successively obtained tables, without any alteration of the rules, till no more negative g_j remain, indicating attainment of a desired maximal solution.

One further iteration of our example gives

TABLE 3.

c_i of basis	Basis	c_j:	0	$-M$	-5	0	38	4	8
		Vector:	$B = A_0$	A_1	A_2	A_3	A_4	A_5	A_6
8	A_6		1	1/6	0	$-1/6$	0	-1	1
-5	A_2		52	3	1	2	0	5	0
38	A_4		8	2/3	0	1/3	1	2	0
$g_j = (\sum_i c_i a_{ij}) - c_j$:			52	$M + 11\frac{2}{3}$	0	4/3	0	39	0

All g_j are now $\geqq 0$, indicating termination. Column A_0 gives

$$x_6' = 1, \qquad x_2' = 52, \qquad x_4' = 8, \qquad x_1' = x_3' = x_5' = 0,$$

so that (6.9) finally gives the desired maximal solution of problem (2.6) as

$$x_1 = 0, \qquad x_2 = 52, \qquad x_3 = 1, \qquad x_4 = 8, \qquad (x_5 = x_6 = 0). \tag{8.8}$$

The maximum profit attainable under the stipulated restrictions is indicated in the final table as $f = g_0 = 52$, which figure can be checked by substituting (8) into (2.8).

It is noteworthy in this example that realization of maximum overall profit necessitates inclusion in the production program of a large quantity of item x_2, although this item, taken by itself, causes the producer to sustain a loss of $1000c_2 = \$5000$ per unit quantity.

9. Interpretation of the g_j. A significant question is: If we were to depart from the maximal solution by replacing $x_k = 0$ by some non-zero (i.e. positive) value for x_k, and adjustment of all other x_i so as still to satisfy inequalities (2.1) and (2.2), what would be the attending decrease in the value of f?

At any stage of iteration, departure from a feasible solution by adjoining the value $x_k = \theta$, and leaving all other conditions satisfied, corresponds to introduction of A_k into the basis with coefficient θ as per (7.1) and (7.2). By (7.5), the decrease in f will then be $f - f' = \theta g_k$. Hence,

Lemma 7. g_j represents the decrease in f per unit increase of x_j (x_j not in the basis).

At every stage, including the final one, the table thus reveals at a glance the effect of departing from the listed basic solution. The g_j of a slack variable not occurring in the final solution will indicate the cost of tightening or relaxing that particular restriction by unity.

Furthermore, solutions "almost as good" as the maximal solution may sometimes be preferred by virtue of extraneous considerations. While the utilization of any one of the otherwise excluded variables will necessitate readjustment of all other variables, the composite effect on f can be determined instantly by a mere inspection of the g_j.

10. Check row. A very convenient check on arithmetic may be obtained by

Rule 9. *Carry the values* $a_{m+2,j} = 1 - \sum_{i=1}^{m+1} a_{ij}$ *as a special row of* A. *As a check, we must have after every matrix transformation* $\sum_{i=1}^{m+2} a'_{ij} - 1 = 0$ *for each* j.

This amounts to the interpretation of the g_j-row as an additional restriction and taking all $c_j = 1$ for checking purposes only.

11. Degeneracy. Whenever, in Rule 7, the smallest positive q_i occurs for several i, then by (7.7) more than one term in (7.2) will become zero, indicating degeneracy. Furthermore, by (7.5), f would show no increase at that stage, which opens the door to the disastrous possibility of "cycling": introducing and eliminating vectors until eventually arriving at a previous basis, whence the cycle is repeated indefinitely. The following "perturbation" treatment resolves the dilemma.

Suppose that at any stage the sequence of matrix columns is $P_1 \cdots P_n$. The unit basis $A_1 \cdots A_m$ no longer coincides with $P_1 \cdots P_m$ but is irregularly distributed over the matrix. Letting ϵ be a small positive number, consider B replaced by

$$B' = B + \sum_{j=1}^{n} \epsilon^j P_j \,, \tag{11.1}$$

where ϵ^j denotes the jth power of ϵ. This modified version will include the original problem by the specialization $\epsilon \to 0$. Using (5.2) for the P_j, (6.7) will be replaced by

$$\left.\begin{aligned} &A_1(x_1 + \epsilon p_{11} + \epsilon^2 p_{12} + \cdots + \epsilon^n p_{1n}) \\ &\quad + A_2(x_2 + \epsilon p_{21} + \epsilon^2 p_{22} + \cdots + \epsilon^n p_{2n}) \\ &\quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &\quad + A_m(x_m + \epsilon p_{m1} + \epsilon^2 p_{m2} + \cdots + \epsilon^n p_{mn}) \\ &\qquad = B + \epsilon P_1 + \epsilon^2 P_2 + \cdots + \epsilon^n P_n \end{aligned}\right\} \tag{11.2}$$

Abbreviating

$$\xi_i = x_i + \epsilon p_{i1} + \epsilon^2 p_{i2} + \cdots + \epsilon^n p_{in} \tag{11.3}$$

(6.8) will be replaced, in view of (7.4), by

$$f = c_1\xi_1 + \cdots + c_m\xi_m = (c_1x_1 + \cdots + c_mx_m) + \epsilon f_1 + \epsilon^2 f_2 + \cdots + \epsilon^n f_n \,. \tag{11.4}$$

As in the derivation of (7.2), the addition of θ times the identity (7.1) to (2) gives

$$A_k \cdot \theta + A_1(\xi_1 - \theta a_{1k}) + \cdots + A_r(\xi_r - \theta a_{rk}) \cdots + A_m(\xi_m - \theta a_{mk}) = B' \tag{11.5}$$

and the subtraction of θ times (7.4) [for $j = k$] from (11.4) replaces (7.5) by

$$f' = f - \theta(f_k - c_k) = c_k\theta + c_1(\xi_1 - \theta a_{1k}) + \cdots + c_m(\xi_m - \theta a_{mk}). \tag{11.6}$$

Suppose rule 7 had produced "ties" for the i − values $i_1 < i_2 < \cdots < i_\nu < \cdots$ and is now applied to (5). For those i which gave $x_{i_1}/a_{i_1k} = x_{i_2}/a_{i_2k} = \cdots$ we now have to compare

$$\begin{aligned} q_{i_1} &= \frac{\xi_{i_1}}{a_{i_1k}} = \frac{x_{i_1}}{a_{i_1k}} + \epsilon\,\frac{p_{i_11}}{a_{i_1k}} + \epsilon^2\,\frac{p_{i_12}}{a_{i_1k}} + \cdots + \epsilon^n\,\frac{p_{i_1n}}{a_{i_1k}} \,, \\ q_{i_2} &= \frac{\xi_{i_2}}{a_{i_2k}} = \frac{x_{i_2}}{a_{i_2k}} + \epsilon\,\frac{p_{i_21}}{a_{i_2k}} + \epsilon^2\,\frac{p_{i_22}}{a_{i_2k}} + \cdots + \epsilon^n\,\frac{p_{i_2n}}{a_{i_2k}} \text{ etc.} \end{aligned} \tag{11.7}$$

Now, for sufficiently small ϵ, P_1 will in general make some of the factors of ϵ in (7) larger than others and thus eliminate the corresponding rows from further competition involving higher powers of ϵ. P_2 will then, in general, by a similar effect of ϵ^2, eliminate further rows from subsequent competition; etc. At no stage can *all* of several remaining tied rows be thus eliminated. On reaching comparison for factors of ϵ^j, whenever P_j is one of the unit columns with its non-zero element in one of the tied rows, say $P_j = I_{i_r}$, ($p_{ij} = 1$ for $i = i_r$, $p_{ij} = 0$ for $i \neq i_r$), if i_r has not already been eliminated by this process, it will now certainly be eliminated. And since all I_{i_r} (for all tied rows i_r) are present among the P_j, *the process must eventually resolve all ties.*

There will thus result a uniquely determined smallest q_i, giving

$$\theta = \min_{r=1,2,\cdots} \frac{1}{a_{i_r k}} (x_{i_r} + \epsilon p_{i_r 1} + \epsilon^2 p_{i_r 2} + \cdots + \epsilon^n p_{i_r n}) \tag{11.8}$$

and among the coefficients $\xi_{i_r} - \theta a_{i_r k}$, which had previously all been equal to zero, there will now be precisely one smallest $= 0$, all others being > 0. In (5), only one of the basis vectors A_{i_r} will now be eliminated. *Degeneracy is thus avoided.*

In case ties are again encountered during a later iteration, it is merely necessary, for all such ties, to take the sequence of P_j, for (3), in the same order $P_1 \cdots P_n$ as before, i.e. again simply from left to right.

It can be shown that no difficulties will be encountered by the use of this procedure if the original unit basis $A_1 \cdots A_m$ is written at the left-hand side of matrix A, through which the sequence of P_j will begin its progression in the case of degeneracy.

In practice, there is no need whatever to carry through the preceding analysis, as the emerging procedure can be simply summarized by

Rule 10. *In the case of ties for r, consider P_t and A_k, starting with $t = 1$. Only for those i which are tied, form the quotients $q_i = p_{it}/a_{ik}$ and take r = the i corresponding to the algebraically smallest q_i (negative p_{it} may now make q_i negative). If ties remain, consider only the i for the remaining ties, increase t by* 1, *and repeat. After r is uniquely determined, continue with the transformation as usual.*

As stated, the presence of m unit vectors guarantees that any ties must be broken before t reaches the value n. Usually $t = 1$ or 2 already does the trick.

12. Special situations. (i) *All* $g_j \geqq 0$. We have seen in Lemma 3 that this represents attainment of a maximal solution.

(ii) *For some negative* g_j, *all corresponding* a_{ij} *are* $\leqq 0$. Eq. (7.5) and Lemma 4 show that we may take $\theta \to \infty$, leading to a feasible solution that includes both x_r and x_k and for which we obtain

Lemma 8. If, for any k, $g_k < 0$ and all $a_{ik} \leqq 0$, then $f_{\max} = \infty$ with $x_k = \infty$.

If desired, this situation can be excluded *a priori*, and finiteness of all variables insured, by adjoining to the original set of inequalities the additional condition $\sum_{i=1}^{n} x_i \leqq N$, where N is a sufficiently large number. If the slack variable x_{n+1} corresponding to this additional restriction appears in the final basis, we shall know that all variables would have remained finite anyhow. If x_{n+1} does *not* appear in the final basis, and some of the other variables x_j are large, we shall know that the growth of one or more variables towards infinity was arrested solely by the present extra restriction.

(iii) $\theta = 0$. This will occur whenever the corresponding x_i in (7.7) is $= 0$. By (6.7), B is then expressed in terms of less than m basis vectors A_i and we again have degeneracy. In this case, from (8.6), $g_0' = g_0$ and $g_k' = 0$ so that f does not increase. Here again,

application of the foregoing perturbation method is equivalent, by (11.8), to making θ non-zero, resulting in a corresponding infinitesimal increase in f. Rules 7, 8, and 10 are applied to this case without modification.

(iv) *Multiple solutions.* On termination, when all $g_i \geqq 0$, we know that no feasible solution can be found for which f would be larger yet. However, it is quite possible that other maximal solutions exist which possess an identical value for f. It may be desirable to determine these alternative maximal solutions.

At first we determine alternative maximal solutions which are *basic.* This corresponds to a matrix transformation as before, with no change in f. By (7.5) and (7.7) this will be possible whenever either

(1) there exist $g_k = 0$ associated with vectors not in the basis (i.e. A_k not a unit column) with $a_{ik} > 0$ for some i; or

(2) one or more values b_r ($= x_r$ in the basis) are $= 0$ with $a_{rj} > 0$ for some j.

With such values of k or r, all possible basic maximal solutions can be obtained by transformation as before. Non-basic maximal solutions can then be obtained as linear combinations of basic maximal solutions. Further solutions that are "almost maximal" with slightly lower f can then be found by Sec. 9.

(v) *The final basis includes an artificial vector with associated variable different from zero.* This result indicates incompatibility of the original set of inequalities.

13. Convergence. At each iteration, by Lemma 2, f increases progressively; hence no solution can reappear at a later iteration. Furthermore, the n column vectors admit at most $\binom{n}{m}$ different bases and, from (6.7), any solution for any one basis must be unique. Hence there can exist at most $\binom{n}{m}$ different basic feasible solutions; and since none can reappear, *our procedure will terminate in a finite number of steps,* either because some $g_i < 0$ with all corresponding $a_{ij} \geqq 0$, indicating by Lemma 8 that $f_{\max} = \infty$, or because all $g_i \geqq 0$, indicating that $f_{\max}$ has been reached. Experience shows that the maximal solution is reached, on the average, in m to $2m$ iterations. It must be emphasized that although no previous *solution* can reappear, it may well happen that individual *vectors* are removed from the basis and later reintroduced; or are introduced into the basis and later on removed again. It is not the presence of any particular basis vector, but the totality of basis vectors, that is non-repetitive.

14. Alternative methods. Two approaches other than the simplex method are available for the solution of linear programming problems. First, it can be shown that an equivalent problem is the determination of the best strategies for playing a "zero-sum two-person game", enabling the application of game theory. Second, successive approximations may be obtained by a relaxation process, for which, however, a very large number of iterations may be required.

For efficient solution, these two methods are of subordinate importance, as experience on the various methods of solution has demonstrated the decisive superiority of the simplex procedure over other methods.

15. The transportation problem. Three typical problems in this category are the following:

(i) Given m sources of a certain commodity to be shipped to n destinations. Source i supplies quantity s_i, destination j requires quantity d_j. Shipment from source i to destination j costs c_{ij} per item and is made for quantity x_{ij}. How many items should be shipped from any source to any destination so as to minimize overall transportation costs?

Mathematically, the problem is: Find x_{ij} which satisfy

$$\left.\begin{aligned} \sum_{j=1}^{n} x_{ij} &= s_i \qquad (i = 1 \cdots m) \\ \sum_{i=1}^{m} x_{ij} &= d_i \qquad (j = 1 \cdots n) \\ \Big(\sum_i s_i &= \sum_j d_j\Big) \\ \text{all} \qquad x_{ij} &\geqq 0 \\ f = \sum_i \sum_j c_{ij} x_{ij} &= \text{minimum} \end{aligned}\right\}. \tag{15.1}$$

(ii) Given n individuals for n jobs. Individual i obtains rating a_{ij} to do job j. What personnel assignments will give maximum overall efficiency?

(iii) m items are to be produced at n factories with given capacities. The cost for item i at factory j is c_{ij} . What quantities x_{ij} of each item should be produced at each factory so as to minimize total production costs?

Equations (1) might be solved by the simplex method as $m + n - 1$ independent linear conditions on mn unknowns. But as it represents a very special case of (2.1), (2.2), (2.3), far simpler special methods have been developed for its solution (see for example Ref. [3]).

16. Acceleration of computations. A modification of the simplex procedure has been put forward by several authors that will considerably reduce the overall computational effort required.

If we define, for the tth iteration step,

$${}_t y_i \begin{cases} = \dfrac{1}{a_{rk}} & \text{for} \quad i = r \\ = -\dfrac{a_{ik}}{a_{rk}} & \text{for} \quad i \neq r \end{cases} \tag{16.1}$$

and the $m \times m$ matrix

$${}_t E = \begin{pmatrix} 1 & & & & {}_t y_1 & & & \\ & 1 & & & {}_t y_2 & & & \\ & & 1 & & \cdot & & & \\ & & & \cdot & \cdot & & & \\ & & & & \cdot & & & \\ & & & & \cdot\cdot & & & \\ & & & & {}_t y_r & \cdot & & \\ & & & & \cdot & & \cdot & \\ & & & & \cdot & & & \cdot \\ & & & & \cdot & & & & 1 \\ & & & & \cdot & & & & & 1 \\ & & & & {}_t y_m & & & & & & 1 \end{pmatrix} \tag{16.2}$$

it will be seen that the transformation (8.4) for the tth iteration may be written

$$A' = {}_tEA. \tag{16.3}$$

For iterations $1, 2, \cdots t$, successive application of (3) shows that

$$A' = ({}_tE\ {}_{t-1}E \cdots {}_2E\ {}_1E)A. \tag{16.4}$$

At this stage, the columns of the original identity submatrix (basis) I will have been converted into the new $m \times m$ submatrix V, say. The new identity submatrix (current basis) J now consists of columns which were, in the original system, submatrix U, say. Other columns, which originally formed submatrix P, will have been converted into Q. We can therefore write, original $A = (I, U, P)$, current $A' = (V, J, Q)$, where the transformation has resulted in

$$I \to V, \qquad U \to J, \qquad P \to Q. \tag{16.5}$$

In this notation, several columns of A may be recorded twice, once under I and once under U; correspondingly for V and J. Similarly, P (or Q) need not be restricted to columns other than I, U (or V, J); any statement for P and Q will include columns I, U and V, J as special cases.

By (4), every column of A is transformed in one and the same manner by matrix premultiplication. Equation (5) therefore gives

$$V = \left(\prod_{\tau=1}^{t} {}_\tau E\right)I, \tag{16.6}$$

$$J = VU, \tag{16.7}$$

$$Q = VP. \tag{16.8}$$

J is the current identity submatrix with scrambled column order. Denoting the transpose of J by J^T, we have $JJ^T = J^TJ = I$. Combination of these relations with (7) shows that

$$\begin{aligned} U &= V^{-1}J, \qquad U^{-1} = J^TV, \\ V &= JU^{-1}, \qquad V^{-1} = UJ^T, \end{aligned} \tag{16.9}$$

i.e. after appropriate rearrangement of columns or rows, U and V give the inverses of each other.

Let

V^i = ith row of V
P_j = jth column of P
q_{ij} = typical element of Q
g^Q = the row of g_i underneath matrix Q
g_j^Q = the element of row g^Q in column j.

If V is kept up to date, there is no need to compute Q for each iteration, as the final Q on termination of the procedure will follow from (8) as $q_{ij} = V^iP_j$. It is on this crucial fact that the modified simplex method is based.

In the sequel we shall denote a zero row by $0'$, a zero column by 0.

We have seen from (8.4) and (8.6) that g_i may be treated as the $(m+1)^{st}$ component of A_i. On defining ${}_tE'$ as the $(m+1) \times (m+1)$ matrix corresponding to (2), the

overall transformation from A to any A' (now of $m + 1$ rows each) may be taken as equivalent to

$$A' = \left(\prod_{\tau=1}^{t} {}_{\tau}E'\right)A = \begin{pmatrix} V & 0 \\ g^V & 1 \end{pmatrix}A, \tag{16.10}$$

where, on the right, V has been bordered by row g^V at the bottom and by the $(m + 1)^{st}$ unit column on the right.

The cumulative effect of any number of transformations on the entire simplex table may therefore by represented by the following product of partitioned matrices,

$$\begin{pmatrix} V & J & Q \\ g^V & 0' & g^Q \end{pmatrix} = \begin{pmatrix} V & 0 \\ g^V & 1 \end{pmatrix}\cdot\begin{pmatrix} I & U & P \\ 0' & g^U & g^P \end{pmatrix}. \tag{16.11}$$

It is seen from this representation that $g^Q = g^V P + g^P$. Hence, for column j, the formula

$$g_j^Q = g^V \cdot P_j + g_j^P \tag{16.12}$$

furnishes the values of all g_j .

For any one iteration, only the following quantities need by evaluated:

Rule 11. (i) Given V, Q_0 , and all g_j , determine k as usual.

(ii) Obtain only column k from $q_{ik} = V^i P_k$ ($i = 1$ to m).

(iii) Determine r as usual.

(iv) Transform V and g^V as usual by (8.4) or the $(m + 1)$ dimensional form of (16.3).

(v) Obtain only the new column Q_0' from $q_{i0}' = V^i P_0$ ($i = 1$ to m).

(vi) Obtain all new g_j from $g_j^Q = g^V P_j + g_j^P$.

(vii) Repeat.

At the end of the procedure, the complete final matrix Q may be obtained from (11).

The modified method has the following advantages over the regular simplex procedure:

(i) Each iteration step requires transformation only of P_k , V, g^V, Q_0 , and the row of g_j , but *not* of Q (and U). This generally amounts to a considerable saving of labor.

(ii) No reliance need be placed on data that bear accumulated round-off errors. g^Q, Q_0 , P_k and Q are obtained by an application of V and g^V to the original data. And any accumulated round-off errors in V and g^V may be eliminated at any desired stage by recalculation by the $(m + 1)$ dimensional analog of (9), viz.

$$\begin{pmatrix} V & 0 \\ g^V & 1 \end{pmatrix} = \begin{pmatrix} J & 0 \\ 0' & 1 \end{pmatrix} \text{ times the inverse of } \begin{pmatrix} U & 0 \\ g^U & 1 \end{pmatrix}. \tag{16.13}$$

(iii) For each transformation, continued use is made of the large blocks of zeros usually present in the original matrix, which would otherwise rapidly fill up.

17. Determinants. From (16.2) and (16.1),

$$|\ {}_{t}E'\ | = \frac{1}{{}_{t}a_{rk}}$$

so that (16.6) gives

$$|\ V\ | = \frac{1}{\prod_{\tau} {}_{\tau}a_{rk}}$$

and by (16.7),

$$| U | = \frac{| J |}{| V |} = | J | \cdot \prod_{r=1}^{t} {}_r a_{rk} , \qquad \text{where} \qquad | J | = \pm 1. \tag{17.1}$$

Hence the determinant of U is given in magnitude by the product of the pivotal elements, always non-zero and positive, arising in the simplex procedure. The non-singularity of successive $| U |$ is to be expected, since the final basis arises from a non-singular overall transformation of linearly independent vectors.

18. Matrix inversion and simultaneous linear algebraic equations. Given A for inversion, or $Ay = b$ for solution, let $s_i = \sum_j a_{ij}$ and e = a column of m components, each equal to 1.

We first set up the problem

$$Ax = s; \qquad x \geqq 0 \tag{18.1}$$

which we know has the unique, finite, non-negative solution $x = e$. To render all $s_i \geqq 0$, multiply appropriate equations through by -1. This corresponds to premultiplication by a matrix I', which is an identity matrix with plus or minus "ones" on the diagonal. Denoting the changed matrices by primes,

$$A' = I'A \tag{18.2}$$

and we have the problem

$$A'x = s' \qquad (s' \geqq 0) \qquad x \geqq 0. \tag{18.3}$$

From (16.5) and (16.9), we can accomplish the inversion of A' $(= U)$ if matters can be so arranged that the final basis J will involve precisely the m columns of A' and no others. This end result may be forced on the problem at the outset as follows,

(i) Augment A' by a complete identity submatrix I of artificial vectors.

(ii) Require $f = \sum c_i x_i$ = maximum.

(iii) Specify $c_i \begin{cases} = 1 \text{ (say) for columns of } A' \\ = -M \text{ (a large negative number) for all other columns.} \end{cases}$

As explained earlier (Sec. 6), f then cannot possibly reach its maximum as long as any variable x_i with $c_i = -M$ still remains in the basis. Since $x = e$, we know that the maximum of f exists and is finite $(= m)$. The simplex procedure applied to the present linear programming problem must therefore necessarily lead to a final basis involving precisely the m columns of A', so that the entire process can be represented as $(s', A', b', I) \rightarrow (e, J, p, V)$ where, in particular,

$$s' \rightarrow e, \qquad A' \rightarrow J, \qquad b' \rightarrow p, \qquad I \rightarrow V \tag{18.4}$$

and where, by (16.9) and (2), $V = JA'^{-1} = JA^{-1}I'$, whence

$$A^{-1} = J^T V I'. \tag{18.5}$$

Similarly $p = Vb' = JA^{-1} I'\ (I'b)$, whence

$$y \equiv A^{-1}b = J^T p. \tag{18.6}$$

For pure matrix inversion, column b' is omitted.

These products do not require actual matrix multiplication. Premultiplication by J^T merely amounts to a rearrangement of rows, while post-multiplication by I' merely changes the signs of certain columns.

If each iteration step introduces into the basis a column of A' and removes from

the basis a column not in A', then the simplex process requires approximately the same number of operations as ordinary elimination. Selection of k and r corresponds, in elimination, to proper choices for avoiding very small divisors.

The simplex procedure of linear programming may thus be said to represent a generalization of matrix inversion, which it includes as a special case.

19. Duality. In matrix form, we have so far treated the "direct" problem

$$Ax \leqq b, \qquad x \geqq 0, \qquad f \equiv c \cdot x = \text{maximum}. \tag{19.1}$$

A related problem, called the "dual" of (1), reads

$$A^T y \geqq c, \qquad y \geqq 0, \qquad \phi \equiv b \cdot y = \text{minimum}, \tag{19.2}$$

where A^T is the transpose of A; A is of order $m \times n$; x, c are $n \times 1$; and y, b are $m \times 1$.

It can be shown that finite solutions exist either for both or for none of these two problems and when they exist,

$$\max f = \min \phi. \tag{19.3}$$

Solution of (1) simultaneously provides the solution of (2). Using the notation of the preceding section, the values y_i are contained in the final optimal table for (1) as

$$y = g^V. \tag{19.4}$$

If the direct problem is solved with the aid of artificial vectors and the solution y for the dual is desired, the row of g_j should be split into two distinct rows: the first, to list the coefficients of M; and the second, to record values free of M. In the final optimal table for (1), Eq. (4), giving the values of y, then refers to the second g_j row only.

If the system (1) contains several inequalities of the form (3.2), the corresponding dual (2) may be obtained as follows. Multiply such reversed inequalities through by -1. All inequalities are now in the form (1), which may then be translated into its corresponding dual problem.

Any equality $\sum a_{rj}x_j = b_r$ may be converted into inequality form by replacing it by the pair of inequalities $\sum a_{rj}x_j \leqq b_r$ together with $\sum a_{rj}x_j \geqq b_r$ i.e. $\sum (-a_{rj})\, x_j \leqq -b_r$. Each equation j of the corresponding dual will be of the form

$$\sum_{i \neq r} y_i a_{ij} + (y_{r_1} - y_{r_2}) a_{rj} \geqq c_j .$$

This may be considered to represent the dual of the original matrix (a_{ij}), where y_r has been replaced by an "unrestricted" variable $y_r' = y_{r_1} - y_{r_2}$ that may be positive or negative. Thus an equation of the original "direct" system corresponds to an unrestricted variable in the "dual".

A direct problem that permits some of the x_j to be unrestricted (i.e. to take on positive as well as negative values) may be converted to the standard form (1) by the substitution $x_k = x_{k_1} - x_{k_2}$ ($x_{k_1} \geqq 0$, $x_{k_2} \geqq 0$). Matrix A will then contain two identical columns with opposite signs throughout the transformations. Since they are linearly dependent, only one of them can be in the basis at any one time and only one need be recorded explicitly. Corresponding to the columns k_1, k_2, the dual will contain the two inequalities $\sum y_i a_{ik} \geqq c_k$ and $\sum y_i\, (-a_{ik}) \geqq c_k$, whence $\sum y_i a_{ik} = c_k$. Hence an unrestricted variable x_k of the direct problem corresponds, in the dual, to an equation instead of an inequality.

Whenever most inequalities of a problem are of the form (3.2), or whenever the number of restrictions greatly exceeds the number of variables, the matrices involved

in the calculation may be materially reduced by replacing such problem by its dual, possibly making use of (2.5) or its converse.

20. Partitioned problems. Many problems arising in applications result in matrices primarily consisting of several compact blocks of coefficients with blanks in the remainder of the corresponding rows. As a tiny example, consider the problem

TABLE 4.

c_j :	1	1	3	1	1	1	$-M$			
	A_1	A_2	A_3	A_4	A_5	A_6	A_7			A_0
	1	0	1				(0)	x_1	$\leqq$	2
	−1	1	3				(0)	x_2	$\leqq$	3
				0	1	−2	(0)	x_3	$\leqq$	4
				1	−1	7	(0)	x_4	$\leqq$	5
	1	1	1	1	1	1	(−1)	x_5	$\leqq$	16
								x_6		
								(x_7)		

(20.1)

where all $x_j \geqq 0$ and $f = \sum c_j x_j$ = maximum. $A_7 x_7$ is not part of the problem but is included merely for the discussion below.

We may first of all solve the smaller, mutually independent problems given by the individual blocks of submatrices:

Problem 1.

A_1	A_2	A_3	
1	0	1	$\leqq 2$
−1	1	3	$\leqq 3$

(20.2)

Problem 2.

A_4	A_5	A_6	
0	1	−2	$\leqq 4$
1	−1	7	$\leqq 5$

(20.3)

On introduction of slack variables, the simplex procedure applied to each of these two independent problems furnishes the partial maximal solutions

$$x_1 = 2; \quad x_2 = 5; \quad x_3 = 0; \quad x_4 = 9; \quad x_5 = 4; \quad x_6 = 0. \tag{20.4}$$

For partial Problem 1, the basis vectors are thus given as A_1 and A_2. The remaining vector (or vectors) follow from application of (5.1) to (2)

$$\begin{pmatrix} 1 \\ 3 \end{pmatrix} = d_{13} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + d_{23} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

whence

$$A_3 = 1 \cdot A_1 + 4 \cdot A_2 \tag{20.5}$$

and for the slack vectors,

$$\begin{pmatrix}1\\0\end{pmatrix} = s_1 = A_1 + A_2 , \tag{20.6}$$

$$\begin{pmatrix}0\\1\end{pmatrix} = s_2 = 0\cdot A_1 + A_2 . \tag{20.7}$$

In practice, these relations appear explicitly in the final optimal table of the partial problem by virtue of (5.2).

Similarly for (3),

$$A_6 = 5\cdot A_4 - 2\cdot A_5 , \tag{20.8}$$

with slacks

$$\begin{pmatrix}1\\0\end{pmatrix} = s_3 = A_4 + A_5 , \tag{20.9}$$

$$\begin{pmatrix}0\\1\end{pmatrix} = s_4 = A_4 + 0\cdot A_5 . \tag{20.10}$$

The values (4) are now taken as a starting solution of (1). Since $\sum x_i = 20 > 16$, the present x_i can be made to satisfy the last inequality of (1) only on introduction of an artificial vector A_7 as shown in the table, with an associated artificial variable $x_7 \geqq 0$.

A_3 considered as a 2-component vector requires satisfaction of (5). As a 5-component vector its last non-zero component may be expressed in terms of the additional basis vector A_7. Equation (5) then becomes

$$\begin{pmatrix}1\\3\\0\\0\\1\end{pmatrix} = 1\begin{pmatrix}1\\-1\\0\\0\\1\end{pmatrix} + 4\begin{pmatrix}0\\1\\0\\0\\1\end{pmatrix} + 4\begin{pmatrix}0\\0\\0\\0\\-1\end{pmatrix},$$

i.e.

$$A_3 = 1\cdot A_1 + 4\cdot A_2 + 4\cdot A_7 \tag{20.11}$$

and similarly

$$s_1 = 1\cdot A_1 + 1\cdot A_2 + 2\cdot A_7 , \tag{20.12}$$

$$s_2 = 1\cdot A_2 + 1\cdot A_7 . \tag{20.13}$$

In the same way (8), (9), (10) are augmented to

$$A_6 = 5A_4 - 2A_5 + 2A_7 , \tag{20.14}$$

$$s_3 = A_4 + A_5 + 2A_7 , \tag{20.15}$$

$$s_4 = A_4 + A_7 , \tag{20.16}$$

(4) gives

$$A_0 = 2A_1 + 5A_2 + 9A_4 + 4A_5 + 4A_7 , \tag{20.17}$$

while from (1)

$$s_5 = -A_7 . \tag{20.18}$$

These relations express the vectors of (1) in terms of A_1, A_2, A_4, A_5, A_7 taken as basis.

For the overall problem (1), we now wish to use the latter basis vectors as *unit* basis for the initial solution (4) and therefore write them in the form

$$A_1 = \begin{bmatrix}1\\0\\0\\0\\0\end{bmatrix}; \quad A_2 = \begin{bmatrix}0\\1\\0\\0\\0\end{bmatrix}; \quad A_4 = \begin{bmatrix}0\\0\\1\\0\\0\end{bmatrix}; \quad A_5 = \begin{bmatrix}0\\0\\0\\1\\0\end{bmatrix}; \quad A_7 = \begin{bmatrix}0\\0\\0\\0\\1\end{bmatrix}. \tag{20.19}$$

This change of notation requires corresponding changes in the notation of all other column vectors such that the fundamental "basis relations" (11) to (18) remain unaltered. With notation (19), they become

$$\begin{aligned} A_3 &= \begin{bmatrix}1\\4\\0\\0\\4\end{bmatrix}; \quad s_1 = \begin{bmatrix}1\\1\\0\\0\\2\end{bmatrix}; \quad s_2 = \begin{bmatrix}0\\1\\0\\0\\1\end{bmatrix}; \quad A_6 = \begin{bmatrix}0\\0\\5\\-2\\2\end{bmatrix}; \\ s_3 &= \begin{bmatrix}0\\0\\1\\1\\2\end{bmatrix}; \quad s_4 = \begin{bmatrix}0\\0\\1\\0\\1\end{bmatrix}; \quad A_0 = \begin{bmatrix}2\\5\\9\\4\\4\end{bmatrix}; \quad s_5 = \begin{bmatrix}0\\0\\0\\0\\-1\end{bmatrix}. \end{aligned} \tag{20.20}$$

Equation (1), converted to equalities and transformed to the new starting basis, now takes on the form

TABLE 5

c_j of basis	Basis	c_j : 1	1	3	1	1	1	$-M$	0	0	0	0	0	0
		A_1	A_2	A_3	A_4	A_5	A_6	A_7	s_1	s_2	s_3	s_4	s_5	A_0
1	A_1	1	0	1	0	0	0	0	1	0	0	0	0	= 2
1	A_2	0	1	4	0	0	0	0	1	1	0	0	0	= 5
1	A_4	0	0	0	1	0	5	0	0	0	1	1	0	= 9
1	A_5	0	0	0	0	1	−2	0	0	0	1	0	0	= 4
$-M$	A_7	0	0	4	0	0	2	1	2	1	2	1	−1	= 4
	g_j :	0	0	$-4M$ +2	0	0	$-2M$ +2	0	$-2M$ +2	$-M$ +1	$-2M$ +2	$-M$ +1	$+M$	$-4M$ +20

From here on, a single iteration step yields the final solution $x_1 = 1, x_2 = 1, x_3 = 1, x_4 = 9, x_5 = 4, x_6 = 0, x_7 = 0, f_{max} = 18$. Large problems may thus often be broken down into several smaller ones, for which the maximal solutions can be found comparatively easily. With the aid of the resulting partial optimal solutions, satisfaction of the remaining conditions, that tie the individual sections together, will then usually require only a few iterations of the re-formulated large original problem.

21. High-speed electronic computing. The advent of high-speed electronic computers renders the solution of large-scale linear programming problems economically feasible. A standard program was designed by the writer to carry-through the unmodified simplex procedure on the IBM 701, and is now used successfully. Data are handled in floating-decimal form, eliminating scaling analysis for each new problem as well as the common incidence of machine difficulties resulting from unforeseeable fluctuations in the size of numbers. With the present program, any size matrix (after augmentation by slack and artificial vectors) up to 50×100 can be accommodated. In cases of degeneracy, ties are resolved automatically by the procedure of Rule 10. Numerous checks are built into the program, which render undetected machine errors virtually impossible. The time required for an $m \times n$ matrix (all operations in floating-decimal form) is about $(0.5/1000)\ mn$ minutes per iteration.

The time-saving features in case of error detection may be noteworthy. Whenever one of the checks detects a calculation error, the program automatically restores certain data, depending on the subprogram concerned, skips back and tries again. If corrected on the second try, automatic continuation is assured without stopping. On persistence of an error, or on indication of infinite solutions, or on reaching the optimal solution, or if the number of completed iterations has reached a pre-assigned bound, the program automatically prints out the matrix for the last successfully completed iteration together with an identification of the nature and location of the condition that caused print-out.

The value of f for each iteration is likewise printed out to display an indication of the progress of convergence.

Equation (8.4) stipulates special treatment for the rth element of each column. When the transformation is carried out columnwise, this would require a special test for each i to determine whether or not it equals r. The elimination of such a test for each element at the cost of a few additional operations per column would contribute to greater speed. A little artifice was found to accomplish this purpose: at the start of the iteration, column k and the iteration divisor $a_{rk} = d$ are set aside and *the element a_{rk} of the transferred column k is replaced by zero.* Transformation of any column j then proceeds thus: (i) replace a_{rj} by $a'_{rj} = a_{rj}/d$; (ii) replace a_{ij} by $a'_{ij} = a_{ij} - a'_{rj}a_{ik}$ *for all* i. It is readily seen that this will automatically give the correct result for all i including $i = r$ and extend uniformity so far—should this be desired—as to include the case $j = k$ (the non-transferred unaltered column k) both for $i \neq r$ and for $i = r$.

22. Application to optimum gasoline blending. Application of Linear Programming to an industrial problem of wide scope is found in the determination of optimal blends of gasoline components to maximize profit within the framework of meeting specifications on technical properties and availability. The imposed constraints primarily stipulate ranges for vapor pressures, volatilities and octane numbers at various temperatures for the various grades of finished gasoline in terms of data given for the components, as well as supply limitations on each of the components.

As a typical instance, a problem expressed by an 18×34 matrix was solved by

machine in 29 iterations within 20 minutes. As is common with problems solved by means of Linear Programming, the maximum realizable profit was shown to be greater than the yield of past operations.

Conclusion. The impressive power of Linear Programming holds the promise of a significant contribution to the economic machinery of human society.

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ON THE GROWTH OF TAYLOR-GÖRTLER VORTICES ALONG HIGHLY CONCAVE WALLS*

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1. Abstract. The primary objective of this study has been to prepare a chart for computing the growth of Taylor-Görtler vortices in laminar flow along walls of both high and low concave curvature. Taylor-Görtler vortices are streamwise vortices having alternate right- and left-hand rotation that may develop in the laminar boundary layer along a concave surface.

The equations of motion are derived anew and re-examined with regard to the importance of the various terms. The final equations used in preparation of the chart are found to be valid for radii of curvature as small as 30 times the boundary layer thickness. Furthermore, it is shown that the equations are not restricted in validity to cases of constant wall curvature, constant free stream velocities, or to boundary layers of constant thickness. Whereas the previous analyses by Taylor and Görtler assumed the vortex to grow exponentially as a function of time, the present study recasts the growth into a more convenient form in which the vortex grows as a function of distance.

The solution is an eigenvalue problem, which in the present study has been solved mainly by Galerkin's method—a variational method. Both the eigenvalues and the eigenfunctions are presented, the former in the aforementioned chart. It is possible to compare the solutions for neutral stability with those given by Görtler. The two solutions are in approximate agreement.

A second method of solution also is described. This method is believed to offer considerable improvement, provided a high-speed digital computer is available. In the one case checked by both methods agreement was within 2%.

Finally, the stability chart was applied to all the known experimental data concerning the effect of concave curvature on the transition point. The well known parameter $R_\theta(\theta/r)^{1/2}$ is shown to be inadequate as an indicator of the transition point. Instead, the experimental data indicate that an apparent amplification factor, $\exp \int \beta\, dx$, is a much better measure. Available results show that transition of this type will occur when $\int \beta\, dx$ reaches a value of about ten.

2. The flow past a concave plate. A considerable body of indirect evidence indicates that a laminar flow along a concave wall does not remain two-dimensional. Instead, it rolls up into alternate right- and left-hand vortices as indicated in Fig. 1. To obtain some insight into the forces that cause the formation of these vortices, consider the streaming of an incompressible, viscous fluid past a concave wall, Fig. 2. If the Reynolds number is not extremely low, a boundary layer will develop.

At some arbitrary height, y_1, within the boundary layer, the velocity is u_1. At some other height $y_2 = y_1 + \Delta y$ the velocity is $u_2 = u_1 + (\partial u/\partial y)\Delta y$. At the height y_1, the fluid is under a pressure p_1. By the usual boundary layer equations for two-dimensional

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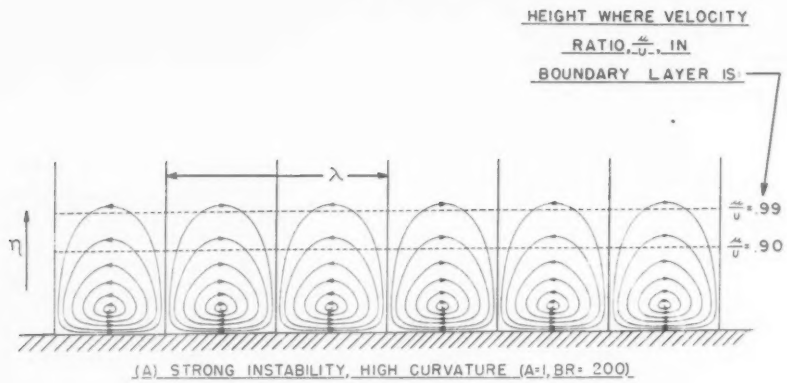
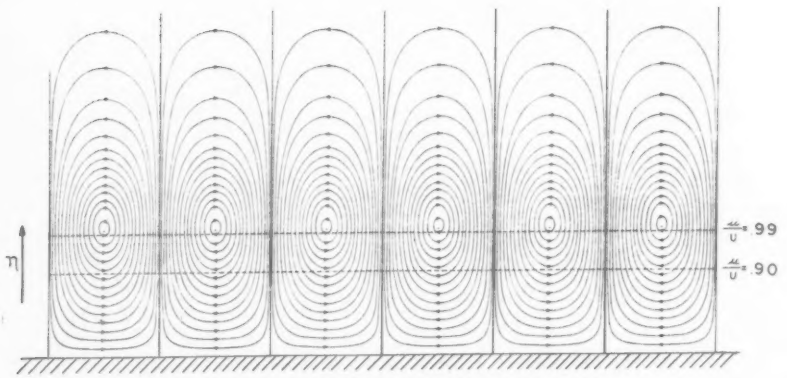


(A) STRONG INSTABILITY, HIGH CURVATURE (A=1, BR= 200)



(B) NEUTRAL STABILITY, SLIGHT CURVATURE (A=1, BR=0)

FIG. 1. Streamline patterns for one width of vortex as calculated for walls of high and low concave curvature. View looking directly upstream or downstream.

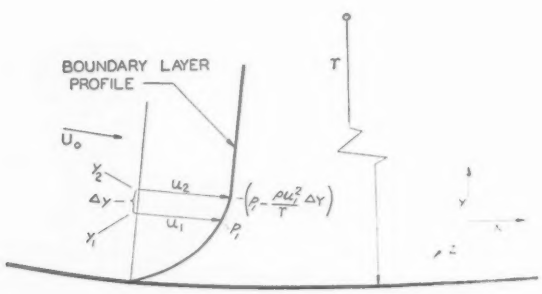


FIG. 2. Boundary layer flow along a curved plate.

curved flow, (for instance, Eq. 8, page 120, Ref. [1]) the steady state pressure at y_2 is $p_1 - (\rho/r)u_1^2\Delta y$.

Suppose an element of fluid from the position y_1 is displaced outwards, as by a swirl, until it reaches the height y_2 . If there is negligible energy dissipation, the element will not lose any total head, so that with the pressure existing at y_2 , a new velocity u_2' will be obtained as follows:

$$p_1 + \tfrac{1}{2}\rho u_1^2 = p_2 + \tfrac{1}{2}\rho u_2'^2$$
$$= p_1 - r^{-1}\rho u_1^2\Delta y + \tfrac{1}{2}\rho u_2'^2$$

or

$$u_2' = u_1(1 + 2\Delta y/r)^{1/2}.$$

Expanding, the inviscid change in velocity, $u_2' - u_1$, to first order is $u_1(\Delta y/r)$.

In the basic boundary layer the change in velocity accompanying a change Δy is $u_2 - u_1 = (\partial u/\partial y)\Delta y$. Thus the change $u_2' - u_1$ is smaller than $u_2 - u_1$ by the order of δ/r where δ is the boundary layer thickness. The displaced element of fluid will now be out of balance with its surroundings because of centrifugal force. If the element has been displaced outwards, the centrifugal effect will give it a sort of buoyancy that enables it to continue its movement. If displaced inwards from its position of equilibrium, it will tend to move right to the wall, for its centrifugal force will exceed that of the surrounding fluid. Thus, in the case of concave flow, a condition of instability exists. The situation is somewhat similar to that occurring when a body of fluid rests over a hot surface.

Such a condition of instability in the flow has been recognized since early in this century, Lord Rayleigh [2] being one of the first to study it. His studies, however, were concerned only with the flow between rotating concentric cylinders. Apparently, until G. I. Taylor [3] tackled the problem no one had conceived the type of motion that this instability would produce. Taylor assumed it would produce vortices in the boundary layer which would trail in the direction of flow. In some respects, they are similar to wing trailing vortices except that they have alternate right- and left-hand rotation. Taylor formulated the motion in mathematical terms, analyzed its stability and verified the analysis experimentally in quite conclusive fashion. However, his attentions were concentrated entirely on the problem of flow between rotating concentric cylinders.

The first investigation of the vortex mode of motion in a boundary layer was carried out by Görtler [4] at the Kaiser Wilhelm Institute under Prandtl. Görtler's analysis was exploratory in approach, for, never having been made before, there was no evidence that this type of flow could exist in a boundary layer. In fact, one of the reasons for the investigation was that the group at Göttingen was exploring all avenues that might assist in explaining the onset of turbulence. Görtler's analysis produced a parameter that served as a measure of the instability of the flow, namely, $R_\theta(\theta/r)^{1/2}$ where R_θ is the boundary layer Reynolds number $(U_0\theta/\nu)$, θ the momentum thickness and r the radius of curvature of the plate. Görtler performed no experiments to verify his analysis. Subsequently Liepmann [5] at the California Institute of Technology ran tests which did verify Görtler's analysis that $R_\theta(\theta/r)^{1/2}$ was a controlling parameter. Liepmann's tests were confined to measuring the effect of curvature on transition, no effort being made to ascertain whether vortices of the type assumed in the mathematics did exist

in fact. Nevertheless, the experimental data point very strongly to the existence of this type of vortex flow in the boundary layer. In retrospect:

(a) Taylor and Hagerty [6] show photographs of this type of vortex flow between rotating cylinders.
(b) The stability, size, and shape of the vortices in (a) agree excellently with calculations.
(c) The theoretical parameter $R_\theta(\theta/r)^{1/2}$ characterizing vortex flow in the boundary layer of a concave wall, was demonstrated by Liepmann to be valid.

3. Previous theoretical investigations. Taylor's original analysis in 1923 is not applicable to the problem of flow in a boundary layer, for his work dealt with the stability of viscous flow between concentric rotating cylinders. Furthermore, his equations are formulated in terms of cylindrical coordinates. Several additional theoretical attacks on the same problem were made by other investigators, but not until 1940 was the same mode of disturbance studied for ordinary boundary layer flow—by Görtler.

Görtler's objective was primarily one of learning whether such a type of motion could be unstable. Therefore, to seek the essence of the problem he analyzed only the case of weak instability, for his major concern was the prediction of neutral stability. As a consequence, he was able to drop a large number of terms in the equations and produce a conveniently tractable expression which clearly indicated the dominant parameters. The equations form a sixth order system, but fortunately, they lent themselves well to solution of the eigenvalue problem by means of integral equations using Green's functions. Görtler's work did show that an unstable vortex flow within the boundary layer could exist, but only for plates with concave curvature. His studies indicated that the stability is not very sensitive to the shape of the boundary layer (contrast with Tollmien-Schlichting type of disturbance).

The vortices under consideration represent a steady type of flow. That is, if the vortices were restrained from wandering back and forth in a spanwise direction, a hot wire fixed on the model would show a steady reading. Thus, such a disturbance is quite different from the Tollmien-Schlichting type which produces a wavelike response on the hot wire. Accordingly, it is most logical and nearest to the physical picture to assume that these vortices grow with distance. Görtler, however, as well as Taylor, analyzed a disturbance that grew in time. For their purposes a temporal variation was satisfactory, because the neutral stability condition was the focal point of their work. Obviously, either basis is satisfactory for the determination of neutral stability, because a disturbance that does not grow with time cannot grow with distance. Nevertheless, strictly speaking, a disturbance assumed to be growing in time implies that a hot wire fixed on the surface would show an exponential growth of signal. When dealing with growing boundary layers in a variable velocity field, Görtler's method becomes untenable, for derivatives with respect to both x and t will exist. In the simpler case of parallel flows, the x-derivatives either equal zero or are negligible.

More recently Meksyn [7] solved anew both Taylor's and Görtler's problems by the method of asymptotic expansions. For the problem of stability of motion between two rotating cylinders, his results verify Taylor's calculations, but are obtained by simpler formulae, demanding far less labor to evaluate.

In the case of boundary layer flow, since he begins with Görtler's differential equa-

tions, the problem treated is the limited mathematical one of finding a better and less laborious solution to a given differential equation. However, Meksyn's determination of the neutral stability limit disagrees considerably with Görtler's value, whereas, the results of the present work, obtained by a third and fourth method, agree satisfactorily with Görtler's values. Unfortunately, time was not available to the author to learn the cause for Meksyn's different result*.

4. Purpose of the present work. Regions of concave curvature exist in a number of aerodynamic applications. Compressor blading, turbine blading, and turning vanes are examples of one type. Another type, but beyond the scope of this work, is the isentropic spike diffuser. A third class of examples exists in boundary layer control where velocity discontinuities are used, for immediately behind a discontinuity there exists a region of strong concave curvature. Griffith airfoils are shapes on which such large concave regions occur.

In a special application, regions of strong concave curvature existed, but for very short distances. There was a good chance that even though the flow was strongly unstable, the instability did not last long enough for Görtler vortices to grow sufficiently to cause transition. Bearing in mind that the work of Görtler, Liepmann, and Taylor proves that such vortices dominate the flow in regions of sufficient concave curvature, the problem at hand justified recalculation of the stability of these vortices for large amounts of curvature. The objective was to prepare a chart for use in routine computation. As only the region of least stability was of interest, no new statements about the mathematical properties of the solution can be made. For information regarding the asymptotic nature of the solution, the number of modes in which this instability may occur, etc., the reader is referred to the articles by Görtler, Meksyn, and Hämmerlin. Some caution should be used, however, in examining their limiting solutions in view of the fact that they are obtained from greatly simplified equations.

5. Method of attack. The problem posed is an eigenvalue problem of calculating the rate of growth of Görtler vortices. It is assumed that the vortices never reach great strength. Thus, the equations can be linearized by the usual methods of small disturbance theory. But it is not assumed that the rate of growth is small; in fact, the equations are written to treat cases of great rates of growth. In developing the equations, therefore, many terms had to be retained that Görtler discarded. This action so complicates the equations, that solutions using either Görtler's or Meksyn's method did not seem possible.

At this point a decision became necessary, for two courses were possible: (a) to discard terms until the equations were simple enough to solve by theoretically exact methods, that is, to secure "exact" solutions of an approximate equation, or (b) to retain the second order terms and solve by approximate methods, that is, to secure approximate solutions of an "exact" equation. The second choice was taken in the belief that even the "exact" methods used by Görtler and Meksyn required many approximations, even to solve their simpler problems. In particular, the question of accuracy seemed more easily answerable by using the second course. For instance, the effect of discarded terms is hard to visualize, even when a simple analytic solution is obtainable. However,

*However, the discrepancy as well as the mathematical details of both methods have recently been examined in a Doctoral Dissertation by Günther Hämmerlin [13]. In general, Hämmerlin agrees with Görtler's results.

if approximate methods are used for solving an exact equation, it is certain that the equation itself contributes no error. The error contributed by the approximate method of solution can usually be sensed rather easily by making a few convergence studies.

Of the several approximate methods available, Galerkin's method appeared most promising. It was used successfully, and appears to have given the desired accuracy. Galerkin's method is so powerful for handling linear equations, that the complexity or number of terms has only a minor effect on the total number of man-hours required to obtain an answer. Consequently, once Galerkin's method was forced by requirements of the problem, the incentive to eliminate all terms of secondary importance disappeared. Therefore, in the final equations, only terms of extremely minor effect have been discarded. However, the work was most laborious, for every eigenvalue required the solution of a twelfth order system of simultaneous equations, including the inversion of a difficult twelfth order matrix.

Near the end of the work another method, a much more powerful and more exact method, was developed. This second method requires very little labor to set up a case. If a high speed digital calculator is available, the remaining labor is trivial. Without such a machine, this second method would be even more laborious than Galerkin's. The method is an iterative interpolation procedure that will be described in Sec. 14.

6. Transformation of the Navier-Stokes equation into a system using curvilinear coordinates. Notation in the subsequent sections in general follows that of Refs. [1] and [4]. We shall consider only the case of two-dimensional flow. The basic flow is two-dimensional, but within it will be developed the three-dimensional disturbance. Since the problem under consideration deals with flows along walls of variable curvature, a coordinate system both convenient and capable of accommodating itself to the necessary arbitrary boundaries is an orthogonal system of "parallel" curves. Because the base motion is two-dimensional, the coordinate system will have curvature in only one direction, and the wall itself will be of cylindrical shape. Generatrices of the surface will be at right angles to the plane of the two-dimensional flow. Figure 3 illustrates the

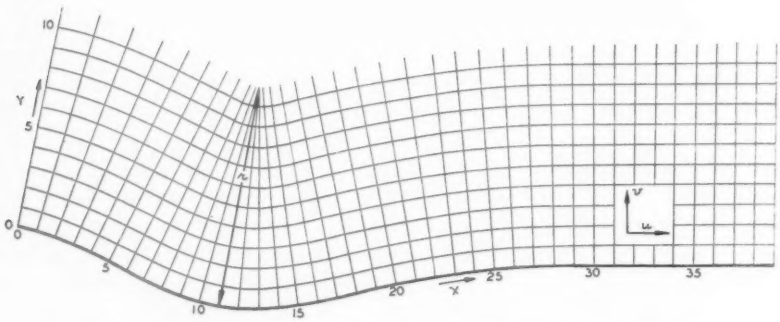


FIG. 3. The parallel curve system of orthogonal curvilinear coordinates.

coordinate system. The line $y = 0$ coincides with the curved wall and the z-direction is perpendicular to the plane of the paper. In the present problem the top of the boundary layer for the most extreme curvature considered would occur about at $y = 1$ in Fig. 3. Therefore, no difficulties with singularities will occur.

In vector notation the steady state incompressible form of the full Navier-Stokes equation is: (Eq. 30, page 100, Ref. [1])

$$\mathbf{v} \times \boldsymbol{\omega} = \operatorname{grad}\left(\frac{p}{\rho} + \frac{1}{2}\mathbf{v}^2\right) + \nu \operatorname{curl} \boldsymbol{\omega}. \tag{1}$$

Following Goldstein, this equation can be translated into the curvilinear coordinate system. The transformation differs in only one respect from that outlined by Goldstein. He assigns a negative sign to concave curvature. In the present analysis, since concave curvature is the center of interest, it is given a positive sign.

The complete, exact Navier-Stokes equations for steady, incompressible flow along a boundary having a curvature $\kappa \equiv 1/r$ are found to be:

$$\left.\begin{aligned}
&\frac{1}{1-\kappa y}u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} - \frac{\kappa uv}{1-\kappa y} = \frac{-1}{\rho(1-\kappa y)}\frac{\partial p}{\partial x} \\
&\qquad + \nu\left[\nabla^2 u - \frac{2\kappa}{(1-\kappa y)^2}\frac{\partial v}{\partial x} - \frac{\kappa^2 u}{(1-\kappa y)^2} - \frac{\kappa}{1-\kappa y}\frac{\partial u}{\partial y} - \frac{v(d\kappa/dx)}{(1-\kappa y)^3}\right] \\
&\frac{1}{1-\kappa y}u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + \frac{\kappa u^2}{1-\kappa y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} \\
&\qquad + \nu\left[\nabla^2 v + \frac{2\kappa}{(1-\kappa y)^2}\frac{\partial u}{\partial x} - \frac{\kappa^2 v}{(1-\kappa y)^2} - \frac{\kappa}{1-\kappa y}\frac{\partial v}{\partial y} + \frac{u(d\kappa/dx)}{(1-\kappa y)^3}\right] \\
&\frac{1}{1-\kappa y}u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left[\nabla^2 w - \frac{\kappa}{1-\kappa y}\frac{\partial w}{\partial y}\right]
\end{aligned}\right\} \tag{2}$$

$$\operatorname{Div}\mathbf{v} \equiv \nabla\cdot\mathbf{v} = \frac{1}{1-\kappa y}\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} - \frac{\kappa v}{1-\kappa y} = 0, \tag{3}$$

where $$\nabla^2 \equiv \frac{1}{(1-\kappa y)^2}\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + \frac{y(d\kappa/dx)}{(1-\kappa y)^3}\frac{\partial}{\partial x}.$$

7. Mathematical description of the disturbance. Görtler, following the work of Taylor, described the vortex motion in the boundary layer by means of the following equations, in which all values are real numbers.

$$\left.\begin{aligned}
u &= u_0(y) + u_1(y)\cos\alpha z \exp\beta t, \\
v &= v_1(y)\cos\alpha z \exp\beta t, \\
w &= w_1(y)\sin\alpha z \exp\beta t, \\
p &= p_0(y) + p_1(y)\cos\alpha z \exp\beta t.
\end{aligned}\right\} \tag{4}$$

In the above: u, v and w are components of the entire velocity as already used; u_0 represents the undisturbed velocity that would exist for the same flow conditions on a plate without curvature; p_0 likewise represents the basic, undisturbed pressure; u_1, v_1, w_1, p_1 are the superimposed disturbance functions; α is a measure of the width of the vortex; $\alpha = 2\pi/\lambda$ where λ is the wave length (see Fig. 1); and β is a measure of the rate of growth of the disturbance as a function of time. As these equations show, Görtler treated the simplified case of constant velocity parallel flows, for $v_0 = 0$, and u_0 and p_0 are independent

of x. Moreover, as brought out earlier, the disturbance grows in time proportional to the term $\exp \beta t$, or, if the equations were to be interpreted literally, the expressions (4) state that at one x, y and z location in a steady base flow, the disturbance is changing strength in an exponential fashion. Furthermore, the term $\exp \beta t$ implies that β is invariant with time, that is, the radius of curvature must be constant.

Part of the purpose of the present analysis is to learn the importance of previously neglected terms in concave flows where both the radius of curvature and boundary layer thickness may be functions of x. Therefore, the stability of vortices in a more general flow will be studied by considering the type of disturbance described below, in which β now has a new meaning:

$$\begin{aligned} u(x, y) &= u_0(x, y) + u_1(y) \cos \alpha z \exp \int \beta(x)\, dx, \\ v(x, y) &= v_0(x, y) + v_1(y) \cos \alpha z \exp \int \beta(x)\, dx, \\ w(x, y) &= \quad 0 \quad + w_1(y) \sin \alpha z \exp \int \beta(x)\, dx, \\ p(x, y) &= p_0(x, y) + p_1(y) \cos \alpha z \exp \int \beta(x)\, dx. \end{aligned} \tag{5}$$

Equations (5) assume that u_0 and p_0 may be functions of x, and also assume a vertical component, v_0, of the basic velocity. The vortex is assumed to grow in strength by the factor $\exp \int \beta\, dx$, that is, to grow with distance, not time. Furthermore, if wall curvature varies, β would not remain constant, that is, β varies with x. For this reason, the simpler form used by Görtler is replaced by an integral.

In generalizing the earlier form, an inconsistency is introduced. If, as in Görtler's analysis the base flow is strictly a constant velocity, parallel type along a wall of constant curvature, then u_1, v_1 and w_1 are truly functions of y only. However, when, for instance, the curvature (and hence β) changes, the eigenfunctions u_1, v_1 and w_1 will change. It follows then that these disturbance functions are not strictly functions of y only. Nevertheless, the assumption of independence of x is not believed to introduce appreciable error, for the functions $u_1(y)$, etc., vary only slightly with β. Furthermore, it is shown later that $\partial u_0/\partial x$, $\partial v_0/\partial x$, $\partial \kappa/\partial x$, $\partial \beta/\partial x$, $\partial p_0/\partial x$ are distinctly minor factors compared with the major terms in the equations. This observation is further evidence that neglect of variations with x of u_1, v_1, etc., has an insignificant effect on the solutions.

8. Development of the differential equations. Expressions (5) can be differentiated, as necessary and substituted into Eqs. (2). Although the degree of instability may be great, the strength of the disturbance is assumed to be small. Therefore, products and squares of the disturbance will be neglected with respect to products of the disturbance times base flow values; in other words, the conventional methods of linearization will be followed. After the substitutions have been made, all base flow terms can be eliminated by subtracting Eqs. (2), which, of course, apply equally well to the undisturbed flow. Then we are left with equations from which the factor $\exp \int \beta\, dx \cos \alpha z$ or $\exp \int \beta\, dx \sin \alpha z$ can be extracted and cancelled out to leave a set of ordinary simultaneous differential equations in u_1, v_1, w_1 and p_1, namely, Eqs. (6) and (7)

$$
\left.
\begin{aligned}
&\nu\frac{d^2u_1}{dy^2} - \left(v_0 + \frac{\kappa\nu}{1-\kappa y}\right)\frac{du_1}{dy} + \left[\frac{\nu}{(1-\kappa y)^2}\frac{d\beta}{dx} + \frac{\beta^2\nu}{(1-\kappa y)^2} - \alpha^2\nu\right.\\
&\quad\left. + \frac{\beta y\nu}{(1-\kappa y)^3}\frac{d\kappa}{dx} - \frac{\kappa^2\nu}{(1-\kappa y)^2} - \frac{\beta u_0}{1-\kappa y} + \frac{\partial v_0}{\partial y}\right]u_1 - \left[\frac{\partial u_0}{\partial y} - \frac{\kappa u_0}{1-\kappa y}\right.\\
&\quad\left. + \frac{2\beta\kappa\nu}{(1-\kappa y)^2} + \frac{\nu}{(1-\kappa y)^3}\frac{d\kappa}{dx}\right]v_1 = \frac{\beta p_1}{\rho(1-\kappa y)}\\
&\nu\frac{d^2v_1}{dy^2} - \left(v_0 + \frac{\kappa\nu}{1-\kappa y}\right)\frac{dv_1}{dy} + \left[\frac{\nu}{(1-\kappa y)^2}\frac{d\beta}{dx} + \frac{\beta^2\nu}{(1-\kappa y)^2} - \alpha^2\nu\right.\\
&\quad\left. + \frac{\beta y\nu}{(1-\kappa y)^3}\frac{d\kappa}{dx} - \frac{\kappa^2\nu}{(1-\kappa y)^2} - \frac{\beta u_0}{1-\kappa y} - \frac{\partial v_0}{\partial y}\right]v_1 - \left[\frac{1}{1-\kappa y}\frac{\partial v_0}{\partial x} + \frac{2\kappa u_0}{1-\kappa y}\right.\\
&\quad\left. - \frac{2\beta\kappa\nu}{(1-\kappa y)^2} - \frac{\nu}{(1-\kappa y)^3}\frac{d\kappa}{dx}\right]u_1 = \frac{1}{\rho}\frac{dp_1}{dy}\\
&\nu\frac{d^2w_1}{dy^2} - \left(v_0 + \frac{\kappa\nu}{1-\kappa y}\right)\frac{dw_1}{dy} + \left[\frac{\nu}{(1-\kappa y)^2}\frac{d\beta}{dx} + \frac{\beta^2\nu}{(1-\kappa y)^2} - \alpha^2\nu\right.\\
&\quad\left. + \frac{\beta y\nu}{(1-\kappa y)^3}\frac{d\kappa}{dx} - \frac{\beta u_0}{1-\kappa y}\right]w_1 = -\frac{\alpha p_1}{\rho}
\end{aligned}
\right\} \tag{6}
$$

$$
\frac{1}{1-\kappa y}\beta u_1 + \frac{dv_1}{dy} + \alpha w_1 - \frac{\kappa v_1}{1-\kappa y} = 0 \qquad \text{(continuity)}. \tag{7}
$$

In the development of the equations, $(1-\kappa y)^{-1}\,\partial u_0/\partial x$ has been eliminated by means of Eq. (3) applied to the base flow. Other than for the aforementioned approximations, these equations are exact. The relative importance of the various terms next must be examined. It is more satisfactory to make this examination, however, after the equations have been transformed into a non-dimensional system.

9. Reduction to non-dimensional form. Beyond this point no further use of the original Navier-Stokes equation is needed. Therefore, there should be no conflict in notation if we replace u_1 by $U_0 u$, etc. In particular let

U_0 = free stream base velocity outside the boundary layer
δ = a characteristic measure of the thickness of the boundary layer
c = a measure of the length in the x direction, characteristically a chord length

Then we can write

$$
\begin{aligned}
&R = \frac{U_0\delta}{\nu}, \quad && u_0 = UU_0, && v_0 = VU_0, && w_0 = 0;\\
&B = \beta\delta, && u_1 = uU_0, && v_1 = vU_0, && w_1 = wU_0;\\
&A = \alpha\delta, && y = \eta\delta, && p_1 = C_p\cdot\tfrac{1}{2}\rho U_0^2;\\
&K = \kappa\delta, && \frac{du_1}{dy} = \frac{U_0}{\delta}\frac{du}{d\eta}, && \frac{d^2u_1}{dy^2} = \frac{U_0}{\delta^2}\frac{d^2u}{d\eta^2}, && \frac{d\kappa}{dx} = \frac{1}{c^2}\frac{d(c\kappa)}{d(x/c)}, \quad \text{etc.}
\end{aligned}
$$

The vertical dimension chosen as characteristic of the boundary layer thickness is that used in the Falkner-Skan boundary layer equation, Sec. 54, Ref. [1] and illustrated in

Fig. 4. Using this system if δ corresponds to $Y = 1$, then $\eta \equiv Y$. However, to be consistent with Görtler, η will continue to be used. Notice further that if the base flow is one of the Falkner-Skan family, $dV/d\eta$, etc. are functions of η only, not x.

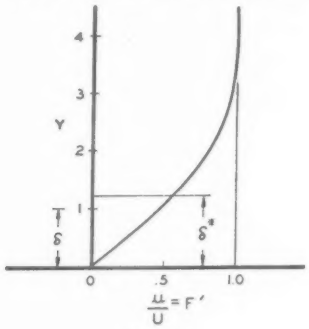


FIG. 4.

Upon making the above substitutions, Eqs. (6) and (7) are transformed into Eqs. (8) and (9).

$$\left.\begin{aligned}
&\frac{d^2u}{d\eta^2} - \left[RV + \frac{K}{1-K\eta}\right]\frac{du}{d\eta} + \left[\frac{(\delta/c)}{(1-K\eta)^2}\frac{dB}{d(x/c)} + \frac{B^2}{(1-K\eta)^2} - A^2\right.\\
&\quad + \frac{(\delta^2/c^2)B\eta}{(1-K\eta)^3}\frac{d(c\kappa)}{d(x/c)} - \frac{K^2}{(1-K\eta)^2} - \frac{BRU}{1-K\eta} + R\frac{dV}{d\eta}\Big]u\\
&\quad - \left[R\frac{dU}{d\eta} - \frac{KRU}{1-K\eta} + \frac{2BK}{(1-K\eta)^2} + \frac{(\delta^2/c^2)}{(1-K\eta)^3}\frac{d(c\kappa)}{d(x/c)}\right]v = \frac{BRC_p}{2(1-K\eta)}\\
&\frac{d^2v}{d\eta^2} - \left[RV + \frac{K}{1-K\eta}\right]\frac{dv}{d\eta} + \left[\frac{(\delta/c)}{(1-K\eta)^2}\frac{dB}{d(x/c)} + \frac{B^2}{(1-K\eta)^2} - A^2\right.\\
&\quad + \frac{(\delta^2/c^2)B\eta}{(1-K\eta)^3}\frac{d(c\kappa)}{d(x/c)} - \frac{K^2}{(1-K\eta)^2} - \frac{BRU}{1-K\eta} - R\frac{dV}{d\eta}\Big]v\\
&\quad - \left[\frac{(\delta/c)}{1-K\eta}R\frac{\partial V}{\partial(x/c)} + \frac{2KRU}{1-K\eta} - \frac{2BK}{(1-K\eta)^2} - \frac{(\delta^2/c^2)}{(1-K\eta)^3}\frac{d(c\kappa)}{d(x/c)}\right]u\\
&\quad = \frac{R}{2}\frac{dC_p}{d\eta}\\
&\frac{d^2w}{d\eta^2} - \left[RV + \frac{K}{1-K\eta}\right]\frac{dw}{d\eta}\\
&\quad + \left[\frac{(\delta/c)}{(1-K\eta)^2}\frac{dB}{d(x/c)} + \frac{B^2}{(1-K\eta)^2} - A^2 + \frac{(\delta^2/c^2)B\eta}{(1-K\eta)^3}\frac{d(c\kappa)}{d(x/c)} - \frac{BRU}{1-K\eta}\right]w\\
&\quad = -\frac{ARC_p}{2}
\end{aligned}\right\} \tag{8}$$

$$\frac{Bu}{1-K\eta} + \frac{dv}{d\eta} + Aw - \frac{Kv}{1-K\eta} = 0. \tag{9}$$

It is desired that the analysis cover approximately the following range of variables.

K = $\delta \div$ radius of curvature—from 0 to .01, that is, to about 30 times the boundary layer thickness,

B = dimensionless growth factor—from 0 to about 0.2,

A = dimensionless measure of vortex size—from 0.1 to 10,

R = $U_0\delta/\nu$—from about 500 to ∞,

$c\kappa$ = airfoil chord $\div$ radius of curvature—from 0 to 10,

$\dfrac{d(c\kappa)}{d(x/c)}$ = dimensionless rate of change of curvature—from 0 to about 200,

$\dfrac{dB}{d(x/c)}$ = dimensionless rate of change of β—from 0 to about 10.

The range of B itself is not preselected; for K is the independent variable. With the range chosen for K this, as well as Görtler's analysis, indicates that B will cover a range about as noted. In physical applications infinite rates of change of curvature, $d(c\kappa)/d(x/c)$ can easily be found. For instance, any discontinuity in curvature is a case; such a discontinuity exists at the pressure jump on a Griffith airfoil. At such places both $d(c\kappa)/d(x/c)$ and $dB/d(x/c)$ exceed the above noted ranges, and under such conditions, the equations to follow will be inadequate. It is not likely, however, that predictions of $\int \beta\, dx$ over a region of concave curvature will be seriously in error, for a strong rate of change of curvature by its very nature, must take place in a short distance.

Orders of magnitude of the various terms were studied carefully in the original work. A tremendous range of magnitudes was found. The largest terms are of the order of R, that is $O(10^3)$. The smallest terms are such as K^2 and $(\delta^2/c^2)B\eta\ d(c\kappa)/d(x/c)$. These terms are of order $< 10^{-4}$ and prove to be of amazingly minor importance, proving that only the magnitude of the curvature is of significance. The fact that K^2 is of the same small order demonstrates that the system of equations can be treated as being essentially linear in K. Certain terms in the equations normally are small, yet at some place in the range they may become the major term. But, examination of the very, very smallest terms shows that at no time do they ever approach dominance; there always exists some other term that is far larger.

The simplified equations are presented below as the system (10) and (11). The only terms dropped are those affecting coefficients by less than 1% at all times, both for the full range of B, K, A and R, and for η less than about 10. To this point no mention of the term $1 - K\eta$ has been made. If $\eta = 10$ and $K = .01$, $K\eta = 0.1$, a value as significant as many terms retained. In Görtler's analysis, the term, $1 - K\eta$, was assumed always to be equal to 1. Consistent with the remainder of the analysis, such an approximation in the present case would not be satisfactory. Therefore, $(1 - K\eta)^{-p}$ is replaced by the first term of its expansion: $1 + pK\eta$. Then if $p = 1$ and $K\eta = .1$, the coefficient is specified correctly within 1%. The transformation not only is a valid approximation mathematically, but physically is likely to be an improvement. In the original development a singularity occurs at $K\eta = 1$. Using the approximate expression, the singularity is removed to $\eta = \infty$. A $(\varphi - \psi)$ net of streamlines and equipotential lines outside the boundary layer will bear little resemblance to the grid system of Fig. 3. Instead, such a net will show a gradual merging into a rectangular system well away from the

surface. A coordinate system in which the ordinates merge at $\eta = \infty$, instead of at $K\eta = 1$, appears actually to be nearer physical reality.

$$\left.\begin{aligned}
&\frac{d^2u}{d\eta^2} - (RV + K)\frac{du}{d\eta} + \left[B^2 - A^2 - BRU + R\frac{dV}{d\eta} - KBRU_\eta\right]u \\
&\quad - \left[R\frac{dU}{d\eta} - KRU\right]v = \frac{BRC_p}{2} \\
&\frac{d^2v}{d\eta^2} - (RV + K)\frac{dv}{d\eta} + \left[B^2 - A^2 - BRU - R\frac{dV}{d\eta} - KBRU_\eta\right]v \\
&\quad - 2KRUu = \frac{R}{2}\frac{dC_p}{d\eta} \\
&\frac{d^2w}{d\eta^2} - (RV + K)\frac{dw}{d\eta} + [B^2 - A^2 - BRU - KBRU_\eta]w = -\frac{ARC_p}{2}
\end{aligned}\right\} \tag{10}$$

$$\frac{Bu}{1 - K\eta} + \frac{dv}{d\eta} - \frac{Kv}{1 - K\eta} + Aw = 0. \tag{11}$$

In solving Eqs. (10) and (11) a certain boundary layer profile, $U(\eta)$, etc., must be specified. If the curvature is gentle, experiment and theory both show that the boundary layer is that predicted for the proper pressure gradients ignoring the curvature. It follows that if the pressure gradient is zero, regardless of the curvature, the boundary layer will be a Blasius profile. However, when the boundary layer thickness is perhaps 1/20 of the radius of curvature, the free stream velocity is undergoing rapid changes, and the boundary profile may not be as expected. But in many cases, θ, the momentum thickness, will be known with reasonable accuracy. Görtler has shown that the instability is mainly a function of θ, not of details of the shape of the boundary layer. As a consequence, the equations should not break down prematurely just because the boundary layer shapes become unknown, provided θ is known.

10. Reduction of Eqs. (10) and (11) to a pair of simultaneous equations. Galerkin's method, used to find the eigenvalue K, requires solution of a large number of simultaneous algebraic equations. If Eqs. (10) and (11) were used as a basis, evaluation of coefficients and the like would be moderately brief, but the number of equations to solve would be excessive. When this system of equations is reduced to two, the time for setup of the simultaneous equations becomes slightly larger, but the number of equations to solve is only two-thirds as many. By this logic, reduction to a single sixth order equation would be best of all. Attempts were made, but unless drastic, accuracy-reducing simplifications are introduced, the equation becomes terribly long, so long that the increased setup time would more than nullify the time saved in solving the simultaneous equations. Hence, a solution beginning with two simultaneous differential equations is believed to require the minimum labor. To avoid introducing errors by the simplifications of the system (10) and (11), reduction was performed by continuing with system (6) and (7). The procedure is lengthy but direct. A system in u and v results. The equations were made non-dimensional and simplified in the fashion used for Eqs. (8) and (9). The resultant "simplified" equations are:

$$\left.\begin{aligned}
&\left(\frac{B^2}{A^2}-1+2K\eta\,\frac{B^2}{A^2}\right)\frac{d^2u}{d\eta^2}+\left\{RV\left(1-\frac{B^2}{A^2}\right)+K\left[1+\frac{B^2}{A^2}-2\eta RV\,\frac{B^2}{A^2}\right]\right\}\frac{du}{d\eta}\\
&\quad+\left\{BRU\left(1-\frac{B^2}{A^2}\right)-R\,\frac{dV}{d\eta}+A^2\left(1-\frac{B^2}{A^2}\right)^2+K\left[BRU\eta\left(1-\frac{3B^2}{A^2}\right)\right.\right.\\
&\quad\left.\left.-RV\,\frac{B^2}{A^2}\right]\right\}u\\
&\quad+\left(\frac{B}{A^2}+\frac{KB\eta}{A^2}\right)\frac{d^3v}{d\eta^3}-\left[\frac{BRV}{A^2}+K\left(\frac{2B}{A^2}+\frac{B\eta RV}{A^2}\right)\right]\frac{d^2v}{d\eta^2}\\
&\quad-\left\{\frac{B^2}{A^2}RU+B\left(1-\frac{B^2}{A^2}\right)+K\left[B\eta\left(1-\frac{3B^2}{A^2}\right)+\frac{2B^2}{A^2}\,\eta RU-\frac{B}{A^2}\,RV\right]\right\}\frac{dv}{d\eta}\\
&\quad+\left\{R\,\frac{dU}{d\eta}-K\left[RU\left(1-\frac{B^2}{A^2}\right)-2B-B\left(1-\frac{B^2}{A^2}\right)\right]\right\}v=0
\end{aligned}\right\}\tag{12a}$$

$$\left.\begin{aligned}
&\left(\frac{B}{A^2}+K\eta\,\frac{B}{A^2}\right)\frac{d^3u}{d\eta^3}-\left[\frac{BRV}{A^2}+K\left(\frac{B\eta RV}{A^2}-\frac{2B}{A^2}\right)\right]\frac{d^2u}{d\eta^2}\\
&\quad-\left\{B\left(1-\frac{B^2}{A^2}\right)+\frac{B^2}{A^2}RU+\frac{B}{A^2}R\,\frac{dV}{d\eta}+K\left[B\eta\left(1-\frac{3B^2}{A^2}\right)+2\,\frac{B^2}{A^2}\,\eta RU\right.\right.\\
&\quad\left.\left.+\frac{B\eta}{A^2}R\,\frac{dV}{d\eta}+\frac{2B}{A^2}RV\right]\right\}\frac{du}{d\eta}\\
&\quad-\left\{\frac{B^2}{A^2}R\,\frac{dU}{d\eta}-K\left[2RU\left(1-\frac{B^2}{A^2}\right)-2\eta\,\frac{B^2}{A^2}R\,\frac{dU}{d\eta}-\frac{B}{A^2}R\,\frac{dV}{d\eta}\right]\right\}u\\
&\quad+\frac{1}{A^2}\frac{d^4v}{d\eta^4}-\left(\frac{RV}{A^2}+\frac{2K}{A^2}\right)\frac{d^3v}{d\eta^3}-\left\{\frac{BRU}{A^2}+\frac{R}{A^2}\,\frac{dV}{d\eta}+2-\frac{B^2}{A^2}\right.\\
&\quad\left.+K\left[\frac{BRU\eta}{A^2}-\frac{RV}{A^2}\right]\right\}\frac{d^2v}{d\eta^2}\\
&\quad+\left\{RV-\frac{BR}{A^2}\,\frac{dU}{d\eta}+K\left[\frac{R}{A^2}\,\frac{dV}{d\eta}-\frac{B\eta}{A^2}\,\frac{R}{}\,\frac{dU}{d\eta}+2+\frac{B^2}{A^2}\right]\right\}\frac{dv}{d\eta}\\
&\quad+\left\{BRU+R\,\frac{dV}{d\eta}+A^2\left(1-\frac{B^2}{A^2}\right)+K\left[BRU\eta+\frac{B}{A^2}R\,\frac{dU}{d\eta}\right]\right\}v=0
\end{aligned}\right\}\tag{12b}$$

If $B = 0$ (Neutral Stability):

$$\frac{d^2u}{d\eta^2}-(RV+K)\,\frac{du}{d\eta}-\left(A^2-R\,\frac{dV}{d\eta}\right)u-\left(R\,\frac{dU}{d\eta}-KRU\right)v=0,\tag{13a}$$

$$\begin{aligned}
2KRUu+\frac{1}{A^2}\frac{d^4v}{d\eta^4}-\left[\frac{RV}{A^2}+\frac{2K}{A^2}\right]\frac{d^3v}{d\eta^3}-\left[\frac{R}{A^2}\,\frac{dV}{d\eta}+2-\frac{KRV}{A^2}\right]\frac{d^2v}{d\eta^2}\\
+\left[RV+K\left(\frac{R}{A^2}\,\frac{dV}{d\eta}+2\right)\right]\frac{dv}{d\eta}+\left(R\,\frac{dV}{d\eta}+A^2\right)v=0.
\end{aligned}\tag{13b}$$

11. Solution by Galerkin's method. Equations (12) can be regrouped as follows:

$$\varphi_1(D)u + \varphi_2(D)v + K[\psi_1(D)u + \psi_2(D)v] = 0,$$
$$\varphi_3(D)u + \varphi_4(D)v + K[\psi_3(D)u + \psi_4(D)v] = 0. \tag{14}$$

Since only K to the first power occurs in the equations, K is the most suitable coefficient to treat as the eigenvalue. B or A could so be treated except that they occur in the equations in a more complicated form. The problem then becomes one of finding the value of K that meets the boundary conditions when A, B, R and the boundary layer profile, $U(\eta)$ are specified. By physical considerations, it is obvious that the disturbance must decay in the free stream. Exactly on the surface there can be no velocity relative to the surface. Therefore, the six boundary conditions are

$$\begin{aligned} u(\infty) &= 0, \qquad u(0) = 0, \\ v(\infty) &= 0, \qquad v(0) = 0, \\ w(\infty) &= 0, \qquad w(0) = 0. \end{aligned} \tag{15}$$

Furthermore, Eq. (11) shows that $dv/d\eta\ (0) = 0$. This last feature may be called a secondary boundary condition. These homogeneous equations with homogeneous boundary conditions have been solved in the main by Galerkin's method. The method is an extremely powerful variation method, in many respects suggestive of the Rayleigh-Ritz method. See Ref. [8] for particulars; only the specific application will be treated in this study. For the present problem, because of the complexity and high order of the equations, six term solutions were used for most computations. They are

$$u = e^{-E\eta}[a_1\eta + a_2\eta^2 + a_3\eta^3 + a_4\eta^4 + a_5\eta^5 + a_6\eta^6], \tag{16a}$$
$$v = e^{-E\eta}[b_2\eta^2 + b_3\eta^2 + b_4\eta^4 + b_5\eta^5 + b_6\eta^6 + b_7\eta^7]. \tag{16b}$$

These automatically meet the six homogeneous boundary conditions. Furthermore (16b) meets the secondary condition, $dv/d\eta(0) = 0$. The decay factor E can be varied as necessary. If properly chosen, an accurate solution can be obtained. If improperly chosen, a poor solution is likely. E is specified, leaving the a_i and b_i as unknowns to be determined.

In the present problem, because calculations were so lengthy, only the Blasius boundary layer was used. Values for it were taken from Ref. [9]. Boundary layer velocity ratios and the various derivatives were approximated to at least three decimal place accuracy by polynomial fits*. The various terms in the Galerkin equations then could be integrated analytically, for only expressions of the form $\eta^p e^{-2E\eta}$ were involved.

12. The matrix problem. Using the Galerkin procedure and expressions (16) twelve simultaneous algebraic equations are obtained, which in matrix algebra can be written:

$$[P + KQ]\{a_1, a_2, a_3, a_4, a_5, a_6, b_2, b_3, b_4, b_5, b_6, b_7\} = 0, \tag{17}$$

where P and Q are square matrices of the coefficients obtained from the Galerkin equations and $\{\ \}$ represents a column matrix in the unknown coefficients. P represents the matrix of coefficients determined from the φ_i expressions, and Q represents the values determined from the ψ_i terms, Eq. (14).

*U, U', U'' etc. were approximated independently; for example, the polynomial approximating U was not differentiated to obtain U'.

The primary interest is to find the mode of flow that will grow at the greatest rate along a wall of specified curvature. If, as in actual calculations, B is prescribed, the problem is to find the least concave curvature that will generate that degree of instability. Since several modes of flow exist, the problem becomes one of finding the least value of K, that is, the largest value of $1/K$, assuming all roots are positive. Accordingly, Eq. (17) is treated in the form

$$\left[\frac{1}{K}P + Q\right]\{a_1 , a_2 , \cdots , a_6 , b_2 , b_3 , \cdots , b_7\} = 0. \tag{17a}$$

Then the roots $1/K$ are given by the characteristic matrix

$$S = P^{-1}Q. \tag{18}$$

The items of major interest are, first, the lowest value of K, and second, the accompanying modal column—the eigenfunction. Because of the high order of the matrix, the iterative method was chosen as being the least laborious. It will converge on the largest value of $| 1/K |$.

13. Accuracy and miscellaneous notes concerning the calculations. Table I lists all cases solved. Most cases were computed using the six-term approximations (16). Approximations using fewer terms were obtained by dropping the highest powers of η in (16). Inspection will show several convergence studies distributed throughout the field which throw light on the accuracy.

The most complete study of convergence was made at $A = 1$, $B = 0$. The very first calculations were made by the method of collocation, Sec. 7.9, Ref. [10]. In this method the error ϵ in a fitted expression is made exactly zero at several points. The collocation method is to Galerkin's method as Lagrangian interpolation is to least squares curve fitting. Ref. [10] states that it is less accurate than Galerkin's method. The present work bears out the statement, for the Galerkin values are always closer to the most exact value when the number of terms in the approximation expressions are equal. The six-term values obtained by both methods agree well, however. The results of the other convergence studies should be clear from the table.

Because the present analysis includes terms neglected by Görtler, the equations cannot be reduced to expressions in terms of his parameters, and computations must be made for specific values of R. Nevertheless, parameters similar to his so predominate that it is convenient to use them. As in his analysis, one parameter is $R_\theta(\theta/r)^{1/2}$, or in the coordinate system of Eqs. (12), $RK^{1/2}$. The other dominant parameter is BR which if referred to momentum thickness would be $\beta\theta R_\theta$. Görtler's similar parameter was $\beta\theta^2/\nu$. Now as $\beta\theta R_\theta \equiv \beta\theta^2 U_0/\nu$ the two differ by a velocity, which follows as a consequence of Görtler's use of a timewise growth of the vortices. To verify that the two parameters $RK^{1/2}$ and BR are dominant, two pairs of cases were checked: $A = .1$, $B = 0$, $R = 2000$ vs $A = .1$, $B = 0$, $R = 6000$, and $A = 3$, $B = .05$, $R = 4000$ vs $A = 3$, $B = .1$, $R = 2000$. If these indeed were the only parameters, then the same value of $RK^{1/2}$ should be obtained for either Reynolds number. The values agree well.

Careful review of all convergence studies made indicates that the individual points appear correct within about two per cent, but such accuracy probably does not hold throughout the final chart, Fig. 5, for the reason that too few points were computed to define well the various curves. The central portions of each curve in Fig. 5 may have errors up to 10 per cent, the extremities even more.

TABLE 1

Summary of Results

A	B	R	E	Method	No. Terms in Approximation	$RK^{1/3}$
.03	.01	2000	2	Galerkin	6	575.6
.10	0	2000	2	"	3	5.10
.10	0	2000	2	"	6	1.353
.10	0	2000	1	"	6	1.430
.10	0	2000	0.5	"	6	0.998
.10	0	6000	0.5	"	6	0.996
.10	.01	2000	2	"	6	186.3
.10	.10	2000	2	"	3	1513.5
.10	.10	2000	2	"	6	890.5
.30	0	2000	2	"	6	1.143
.30	0	2000	1	"	6	1.096
.30	.01	2000	2	"	6	40.54
.30	.05	2000	2	"	6	232.8
.30	.10	2000	2	"	6	500.7
1.0	0	2000	2	"	2	6.530
1.0	0	2000	2	"	3	3.411
1.0	0	2000	2	"	4	2.873
1.0	0	2000	2	"	5	2.865
1.0	0	2000	2	"	6	2.857
1.0	0	2000	2	Collocation	3	1.072
1.0	0	2000	2	"	4	2.624
1.0	0	2000	2	"	6	3.026
1.0	0	2000		Second Method		2.821
1.0	.05	2000	2	Galerkin	6	125.3
1.0	.10	2000	2	"	3	241.6
1.0	.10	2000	2	"	6	241.5
1.0	.20	2000	2	"	3	481.1
3.0	0	2000	2	"	6	16.05
3.0	.05	2000	2	"	6	102.5
3.0	.05	4000	2	"	6	180.3
3.0	.10	2000	2	"	6	178.6
10.0	0	2000	2	"	3	162.7
10.0	0	2000	2	"	6	156.5
10.0	.05	2000	2	"	6	246.7
10.0	.10	2000	2	"	3	327.7
10.0	.10	2000	2	"	6	324.7
10.0	.20	2000	2	"	3	465.8

The disturbance functions, u, v, w, in many cases, are much less accurate. Some cases, by fortuitous choice of E, appear superficially to be good solutions. Others obviously are poor representations of the disturbance. Nevertheless, in spite of the doubtful accuracy of many of the eigenfunctions, most solutions are presented in Figs. 6, 7 and 8,

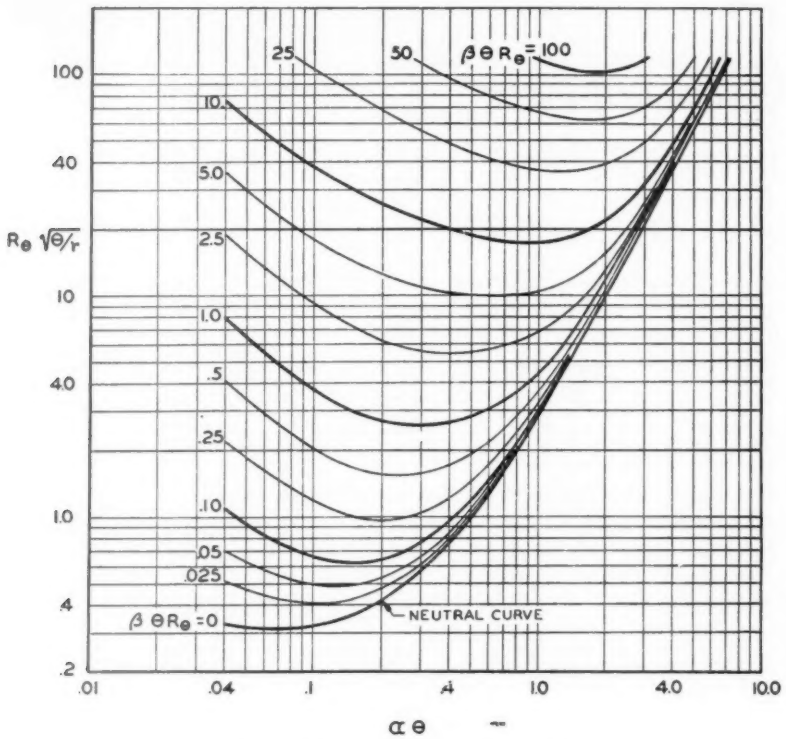


FIG. 5. Stability of Taylor-Görtler vortices.

where some impression of the accuracy may be obtained by comparing the several cases. Further discussion of these functions will be reserved for Sec. 15, Results.

14. A second method of solution. The excessive labor and difficult matrices created a desire to find a faster method of solution—one having improved accuracy if possible. A promising method was found. In any problem of boundary layer flow in free air, the boundary conditions exist at $\eta = 0$ and $\eta = \infty$. But a boundary layer extends only a slight way from $\eta = 0$; beyond it lies the free stream, which may be treated as a region of constant velocity in the direction of η, even on an airfoil. In this region $U = 1$, $dU/d\eta = 0$, etc. Consequently, for problems in boundary layer flow, the disturbance equations simplify to ones in which all important transcendental type of coefficients such as $U(\eta)$ disappear or else become constant coefficients. This fact, together with a matrix interpolation procedure, forms the basis for the method presented on the following pages. The method is far more rapid than the Galerkin method provided a large scale digital computer can be used. Otherwise, it is too slow to be considered as an alternate.

We shall solve the pair of equations (12), although the method would be just as applicable to the original set (10). Equations (12) are a homogeneous pair forming a sixth order system containing an unknown constant. The boundary conditions are homogeneous. In the previous method K was chosen as the unknown because, to the same order of approximation, only K to the first power occurred; whereas B, for instance,

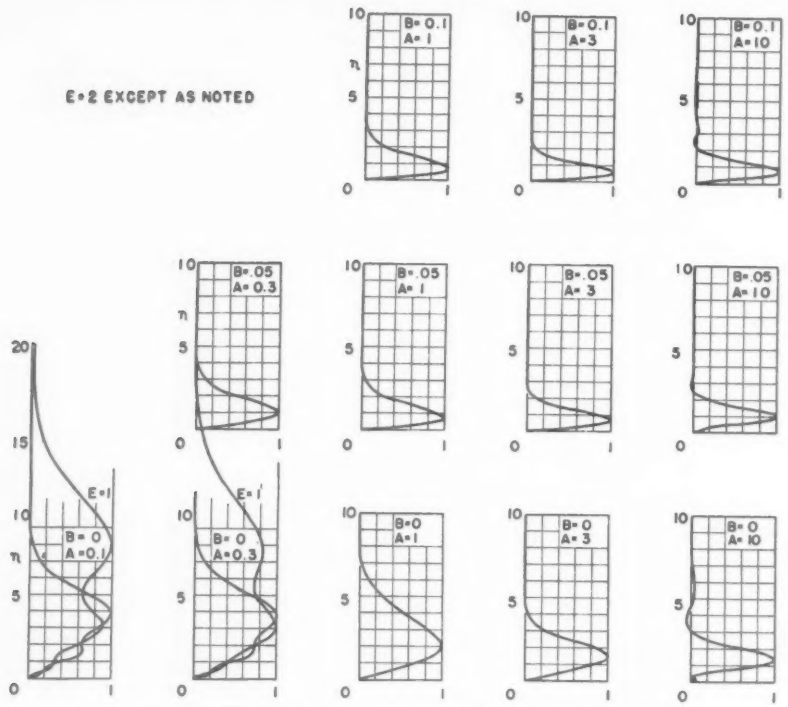


FIG. 6. Nature of the u-component of the disturbance.

occurred in powers as high as B^4. In this new analysis any one of the constants K, B, A, or R could be treated as the unknown. However, K is retained as the unknown, partly because it is convenient and partly because the previous work has been along this line.

The general system is linear but has variable coefficients of a transcendental type, which, however, may be approximated satisfactorily for all practical purposes by simple polynomials. Any homogeneous system of order n has n, and only n, independent solutions, independence being tested by the Wronskian. For the present, assume K is known. Then when six such solutions are found, the complete solution meeting all boundary conditions can be formed. The key to the six can be found by solving the system (12) outside the boundary layer, for beyond the boundary layer, especially the Blasius case

$$RV = \text{Constant} = RV_0\,;\; R\,dV/d\eta = 0;\; U = 1;\; dU/d\eta = 0.$$

In the special case $\beta = 0$ and beyond the boundary layer, the equations reduce to the following pair containing only constant coefficients.

$$\frac{d^2u}{d\eta^2} - (RV_0 + K)\frac{du}{d\eta} - A^2u + KRv = 0,$$

$$2KRu + \frac{1}{A^2}\frac{d^4v}{d\eta^4} - \left(\frac{RV_0}{A^2} + \frac{2K}{A^2}\right)\frac{d^3v}{d\eta^3} - \left(2 - \frac{KRV_0}{A^2}\right)\frac{d^2v}{d\eta^2} \tag{19}$$

$$+ (RV_0 + 2K)\frac{dv}{d\eta} + A^2v = 0.$$

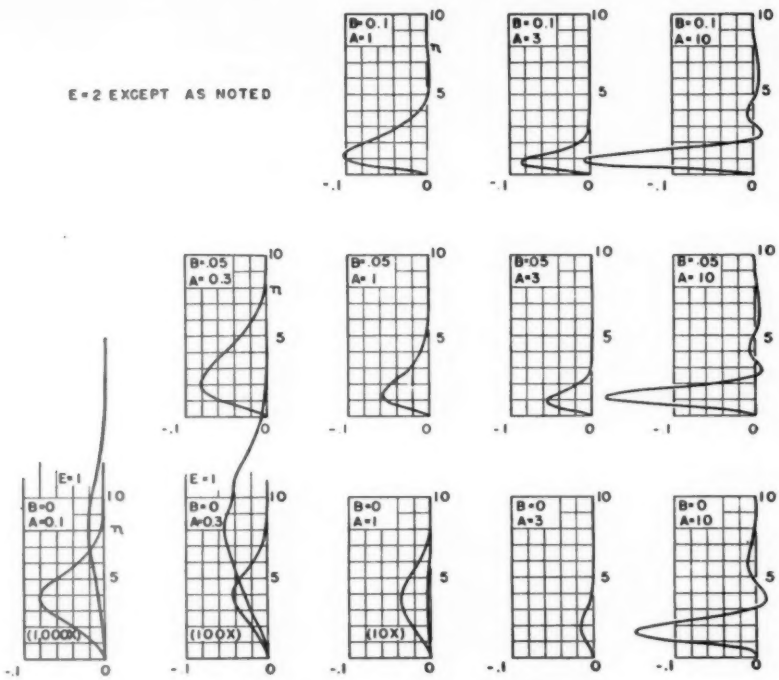


FIG. 7. Nature of the v-component of the disturbance.

Outside the boundary layer, Eqs. (12) can be written in the shorter form

$$(M_1 + N_1K\eta)\frac{d^2u}{d\eta^2} + (M_2 + N_2K\eta)\frac{du}{d\eta} + (M_3 + N_3K\eta)u + (M_4 + N_4K\eta)\frac{d^3v}{d\eta^3}$$
$$+ (M_5 + N_5K\eta)\frac{d^2v}{d\eta^2} + (M_6 + N_6K\eta)\frac{dv}{d\eta} + M_7v = 0$$
$$(P_1 + Q_1K\eta)\frac{d^3u}{d\eta^3} - (P_2 + Q_2K\eta)\frac{d^2u}{d\eta^2} - (P_3 + Q_3K\eta)\frac{du}{d\eta} + P_4u + P_5\frac{d^4v}{d\eta^4} \tag{20}$$
$$- P_6\frac{d^3v}{d\eta^3} - (P_7 + Q_7K\eta)\frac{d^2v}{d\eta^2} + P_8\frac{dv}{d\eta} + (P_9 + Q_9K\eta)v = 0$$

in which M_i , N_i , P_i , Q_i are constants as indicated by Eqs. (12).

Consider the special case $\beta = 0$, (Eq. 19). Six solutions of (19) are readily found, provided K is specified. The solutions obviously are elementary functions, obtained by computing the roots of the sixth degree auxiliary equation. Outside the boundary layer, the general solution of (19) can be written

$$u \text{ or } v = c_1\varphi_1 + c_2\varphi_2 + \cdots + c_6\varphi_6 , \tag{21}$$

where the φ_i are the independent solutions. The c_i differ, of course, according to whether the solutions are for u or v. Because of the way these solutions have been obtained, there

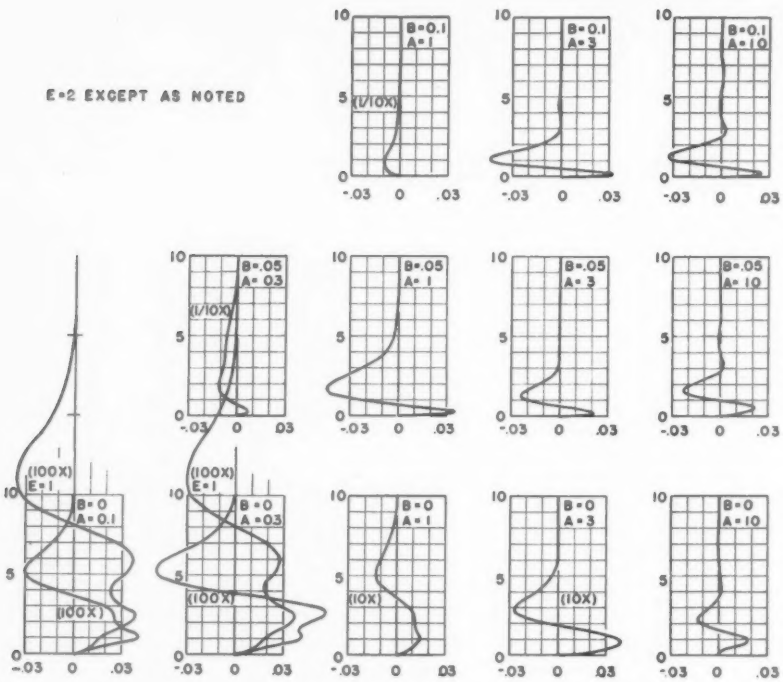


FIG. 8. Nature of the u-component of the disturbance.

is no question of independence. The general solution then for neutral stability is seen to be given in terms of elementary functions for all the region outside the boundary layer. Each complete solution can now be extended through the boundary layer to $\eta = 0$ by means of numerical integration methods. The boundary layer shape can easily be represented with high accuracy. The numerical extension requires only a few minutes on a large scale digital computer. On an IBM Card Programmed Calculator about twenty hours are necessary for each solution φ_i if steps small enough to obtain about four-figure accuracy are used.

Return now to the general case (20). For it, numerical computation within the boundary layer differs only in the number of factors, not in nature. If all initial conditions are clearly specified, there are no problems of uniqueness of the solution within the boundary layer. For consider the general differential equation.

$$p_0(x)\frac{d^n y}{dx^n} + p_1(x)\frac{d^{n-1}y}{dx^{n-1}} + p_2(x)\frac{d^{n-2}y}{dx^{n-2}} + \cdots + p_n(x)y = 0.$$

If the variable coefficients p_0 , p_1 , etc., are continuous, single valued functions of x only, throughout an interval $a \leq x \leq b$ and if p_0 does not vanish at any point in the interval, then there exists a unique continuous solution $y(x)$ which assumes a value of y_0 at a point x_0 within (a, b) and whose first $n - 1$ derivatives are continuous and assume respectively, the values y_0' , y_0'' , $\cdots$, y_0^{n-1} at x_0 . The differential equations (12) have been examined by tracing through what happens to their coefficients when reduced to

a single equation in u or v. Within the boundary layer all the conditions stated above are met. The term equivalent to p_0, the coefficient of $d^6(\ )/d\eta^6$ in the single equation, definitely remains finite in the complete range. Beyond the boundary layer, the solutions are of a different nature due to the existence of the terms $K\eta$. This part of the coefficient of each derivative is small at the joining point because η is < 10, and because of the smallness of K itself ($< .01$). Its small value with respect to the constant term suggests a method of successive approximations, beginning with solutions given by Eqs. (20), but with the terms involving $K\eta$ neglected. One such procedure has been briefly investigated and found suitable although others requiring less labor may well be found.

Within the range of interest it is known that any solution φ_i of (20) may be expressed in a uniformly convergent series in powers of K, a small parameter, Ref. [11], Sec. 3.31.

$$\varphi_i = \varphi_{i0} + K\varphi_{i1} + K^2\varphi_{i2} + K^3\varphi_{i3} + \cdots + K^r\varphi_{ir} + \cdots \qquad (i = 1, 2, \cdots, 6). \quad (22)$$

Because the general system (12) cannot easily be reduced to a single equation, it is convenient to deal in the six paired values u_i and v_i, or

$$\begin{aligned} u_i &= u_{i0} + Ku_{i1} + K^2u_{i2} + K^3u_{i3} + \cdots + K^ru_{ir} + \cdots, \\ v_i &= v_{i0} + Kv_{i1} + K^2v_{i2} + K^3v_{i3} + \cdots + K^rv_{ir} + \cdots, \end{aligned} \qquad (23)$$

where each u_{ir}, v_{ir} is a function of η. Substitute these expressions into (20) and collect terms according to powers of K. An infinite series of linear differential equations is obtained which may be solved in succession to obtain each additional term in (23). Symbolically the system may be written

$$\left.\begin{aligned} L_1(u_{i0}) + L_2(v_{i0}) &= 0 \\ L_3(u_{i0}) + L_4(v_{i0}) &= 0 \end{aligned}\right\} \quad (\text{for } K^0) \quad (a)$$

$$\left.\begin{aligned} L_1(u_{i1}) + L_2(v_{i1}) &= \eta[M_1(u_{i0}) + M_2(v_{i0})] \\ L_3(u_{i1}) + L_4(v_{i1}) &= \eta[M_3(u_{i0}) + M_4(v_{i0})] \end{aligned}\right\} \quad (\text{for } K^1) \quad (b) \qquad (24)$$

$$\left.\begin{aligned} L_1(u_{i2}) + L_2(v_{i2}) &= \eta[M_1(u_{i1}) + M_2(v_{i1})] \\ L_3(u_{i2}) + L_4(v_{i2}) &= \eta[M_3(u_{i1}) + M_4(v_{i1})] \\ \cdots \qquad\quad \cdots &\qquad\quad \cdots \qquad\quad \cdots \end{aligned}\right\} \quad (\text{for } K^2) \quad (c)$$

Solutions u_{i0}, v_{i0} are six in number and involve only the constant part of the coefficients in Eq. (20). They in turn are used in the right-hand side of (24b) which then provides new solutions u_{i1}, v_{i1}. The operators on the left are always the same. Hence, the first correction term reduces to one of determining the particular integral.

Repeated roots never occur in any case of interest. Therefore, the only solutions possible from (24a) are of the type $\exp(\lambda\eta)$, $\exp(\gamma\eta)\cos\mu\eta$, $\exp(\gamma\eta)\sin\mu\eta$. Examination of successive solutions for the particular integrals, by the method of undetermined coefficients, shows that every additional term in the series (23) always contains the same exponential. Thus, if the first approximation, u_{i0}, $v_{i0} \to 0$ as $\eta \to \infty$, any and all successive terms will likewise approach zero. Therefore, it is possible to select solutions capable of meeting boundary conditions $u(\infty)$, $v(\infty)$, $v'(\infty) = 0$, with the certainty that a higher order term in the series will never cause the series to diverge.

In recapitulation, when $B = 0$, initial conditions for the numerical extension of the solutions are given by elementary functions obtained by solving a pair of equations having constant coefficients. When $B \neq 0$, initial conditions can be provided by the solutions (23) of (20).

Of the six independent solutions, only ones having negative exponentials are admissible if the boundary conditions at $\eta = \infty$ are to be met, for no combination of functions involving different positive exponentials will reach a value zero at $\eta = \infty$. Routh's method has been used to examine the signs of roots of the auxiliary equation derived from (24a), that is, of the first approximation. In all cases examined, representing the range of interest, three roots had positive real parts. Therefore, three solutions can be dropped; and for the present problem, Eq. (21) can be written

$$u \text{ or } v = c_1\varphi_1 + c_2\varphi_2 + c_3\varphi_3 \,. \tag{25}$$

For example, in the free stream φ_1 may be $\exp(-\gamma\eta)\cos\mu\eta$, and φ_2 may be $\exp(-\gamma\eta)\sin\mu\eta$. If u_1 corresponds to $\exp(-\gamma\eta)\cos\mu\eta$, then v_1 corresponding will be a combination of φ_1 and φ_2 , as determined by substitution of u_1 in the free stream equations. With these three remaining solutions, all boundary requirements at $\eta = \infty$ are automatically met. For any set of values of B, A, K, R and boundary layer profile $U(\eta)$, numerical extension of the three remaining free stream solutions will provide the following information:

Solution	Numerical values at $\eta = 0$ of:		
φ_1	u_{10}	v_{10}	v'_{10}
φ_2	u_{20}	v_{20}	v'_{20}
φ_3	u_{30}	v_{30}	v'_{30}

The boundary conditions require that $u(0) = v(0) = v'(0) = 0$. Therefore, if q_i are constants

$$\begin{aligned} q_1u_{10} + q_2u_{20} + q_3u_{30} &= 0, \\ q_1v_{10} + q_2v_{20} + q_3v_{30} &= 0, \\ q_1v'_{10} + q_2v'_{20} + q_3v'_{30} &= 0. \end{aligned} \tag{26}$$

In general, this system will be incompatible unless K is chosen correctly. K, however, is an unknown to be found. Assume that the approximate value of K is known, as would be true from previous calculations if a K-A-B plot were being constructed. Since K is known approximately, choose two values for K, preferably bracketing the expected value. Let them be identified as K_1 , K_2 . Compute all six admissible solutions for these two values of K. At $\eta = 0$, intercepts as follows can be read from the results of the numerical integrations.

Solution	Values at $\eta = 0$					
	K_1			K_2		
φ_1	${}_1u_{10}$	${}_1v_{10}$	${}_1v'_{10}$	${}_2u_{10}$	${}_2v_{10}$	${}_2v'_{10}$
φ_2	${}_1u_{20}$	${}_1v_{20}$	${}_1v'_{20}$	${}_2u_{20}$	${}_2v_{20}$	${}_2v'_{20}$
φ_3	${}_1u_{30}$	${}_1v_{30}$	${}_1v'_{30}$	${}_2u_{30}$	${}_2v_{30}$	${}_2v'_{30}$

(27)

If the variation of these values with K is approximately linear, interpolation formulae for each of the nine values can be written, e.g.,

$$u_{10} = {}_1u_{10} + \frac{({}_1u_{10} - {}_2u_{10})}{K_1 - K_2}(K - K_1).$$

Let σ represent the proportionality factor $({}_1u_{10} - {}_2u_{10})/(K_1 - K_2)$ and let $K - K_1 = \Delta K$. Then for u_{10} we can write

$$u_{10} = {}_1u_{10} + \sigma_{11}\Delta K.$$

When all values (27) are used to determine similar relations, Eqs. (26) may be written

$$\begin{aligned} q_1({}_1u_{10} + \sigma_{11}\Delta K) + q_2({}_1u_{20} + \sigma_{12}\Delta K) + q_3({}_1u_{30} + \sigma_{13}\Delta K) &= 0, \\ q_1({}_1v_{10} + \sigma_{21}\Delta K) + q_2({}_1v_{20} + \sigma_{22}\Delta K) + q_3({}_1v_{30} + \sigma_{23}\Delta K) &= 0, \\ q_1({}_1v'_{10} + \sigma_{31}\Delta K) + q_2({}_1v'_{20} + \sigma_{32}\Delta K) + q_3({}_1v'_{30} + \sigma_{33}\Delta K) &= 0, \end{aligned} \tag{28}$$

or in matrix form

$$\begin{bmatrix} {}_1u_{10} & {}_1u_{20} & {}_1u_{30} \\ {}_1v_{10} & {}_1v_{20} & {}_1v_{30} \\ {}_1v'_{10} & {}_1v'_{20} & {}_1v'_{30} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} + \Delta K \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = 0. \tag{29}$$

That is, in short hand notation

$$[P + \Delta KQ]q = 0. \tag{29a}$$

Equation (28) is a cubic in ΔK which may be solved by obtaining the equation explicitly, or by iteration as before for Galerkin's method. If the proper value of K has been bracketed well, the desired root will be the smallest value of ΔK. Otherwise, all roots should be examined. Once the value of $K = K_1 + \Delta K$ is found, a new set of three runs should be made, based on this new-found value, to serve as a check run or to obtain an improved approximation.

Using advanced digital computors, the method is rapid, very accurate, and quite flexible, for either B, A, K or R can be used as the eigenvalue. Furthermore, the method permits the unknown constant to occur in any form. The interpolation procedure will be subject to the same difficulties that beset Newton's method of determining the zeros of a polynomial. Thus, cases may arise where the interpolated K given by (29) may be worse than the earlier trial value. However, such difficulties are not expected to be any more serious than they are in Newton's method. A further drawback of the subject procedure is that it tells nothing about the number and location of other roots, for it obtains only one at a time. If applied to an entirely new problem, exploratory work by other less accurate methods may be desirable.

An example. Only one case has been solved by this method, that for $A = 1$, $B = 0$, $R = 2000$. Solutions were computed for $K = .00001$, and $.000002$. By Galerkin's method, the correct value of K is $.00000204$. Hence, one value is at some distance from the correct value; the other is very close. The three admissible free stream solutions are

$$\varphi_1 = e^{-\omega\eta}, \qquad \varphi_2 = e^{-\lambda\eta}\cos\mu\eta, \qquad \varphi_3 = e^{-\lambda\eta}\sin\mu\eta.$$

For three values of K the usable roots are

K	ω	λ	μ
.0001	1.0235829	.55527247	.14151315
.00001	1.0002648	.56205005	.014604736
.000002	1.0000098	.56213472	.0029290325

The free stream solutions were extended by the Taylor's series method of extrapolation, working directly with Eqs. (13). The joining point chosen as the edge of the boundary layer was at $\eta = 5.7$ corresponding to $U = 1.00000$.

Computations were performed on the IBM Card Programmed Calculator using increments of $\eta = .01$ or .02, according to the nature of the higher derivatives. A few convergence checks using other values of $\Delta\eta$ indicated the solutions were accurate to about three significant figures. By this method, each run requires about twenty hours on the Card Programmed Calculator, that is, 120 hours of computation must be done before a minimum set of values is available for the interpolation procedure. For this reason the runs were mostly of exploratory nature to serve both as a check on the theory back of this second method and to serve as an overall check of Galerkin's method.

15. Results. Because of the great labor involved, a chart for only one Reynolds number-boundary layer combination was prepared. The boundary layer was the Blasius type; the Reynolds number R was 2000, corresponding to $R_\theta = U_0\theta/\nu = 939$. Cross-plots of $RK^{1/2}$ *vs.* B for constant values of A conveniently proved to be nearly straight lines, greatly facilitating interpolation. Recommended interpolation polynomials, representing these cross-plots, are listed in Table 2.

Table 2.

Interpolation formulae for $RK^{1/2}$ as a function of BR and A. $R = 2000$, Blasius boundary layer

A	Formula
0.1	$RK^{1/2} = .998 + 4.4475BR$
0.3	$RK^{1/2} = 1.096 + 1.8581BR + .00598(BR)^2 - .0000139(BR)^3$
1.0	$RK^{1/2} = 2.82 + 1.278BR - .000638(BR)^2 + .000001081(BR)^3$
3.0	$RK^{1/2} = 16 + .9175BR - .000525(BR)^2$
10.0	$RK^{1/2} = 156.4 + .983BR - .000895(BR)^2 + .000000927(BR)^3$

Note: For above case, $\beta\theta R_\theta = (.4696)^2 BR$, $\alpha\theta = .4696A$

$$R_\theta(\theta/r)^{1/2} = (.4696)^{3/2} RK^{1/2}$$

Görtler's work showed the solutions to be primarily a function of θ, the momentum thickness, the actual shape of the boundary layer being found to have only a secondary influence on the results. Therefore, from the cross-plot data, a chart in terms of θ, similar to his, has been constructed, and is presented as Fig. 5. By its use, growth of nearly any size vortex along concave surfaces of any shape can easily be computed simply by

evaluating $\int_{x_1}^{x_2} \beta\, dx$. The chart is most accurate at $R = 2000$ and for the Blasius boundary layer, but as discussed earlier, the parameters used in the plot enable it to be used for other conditions with only a small loss of accuracy. The neutral curve agrees roughly with Görtler's neutral curve. Görtler computed the minimum value of the neutral stability curve to be $R_\theta(\theta/r)^{1/2} = 0.58$; in the present work the value is about 0.32. Meksyn's minimum value was 3.65. Difference between the present and Görtler's values may be attributed to three factors:

(*a*) The present analysis includes the effect of boundary layer growth, that is, $RV \neq 0$.
(*b*) For a given order matrix, Galerkin's method should supply more precise eigenvalues than those given by Green's functions and Fredholm's method of solving the resulting integral equations.
(*c*) Higher order matrices were used in the present study—twelfth order *vs.* eighth for Görtler.

Factors (*b*) and (*c*) are believed to account for most of the disagreement.

Plots of the u, v and w components of disturbances are presented in Figs. 6, 7 and 8. The u disturbance has arbitrarily been given a maximum value of unity; the v and w components then are presented in their true proportions to the u component. The component w is computed by means of Eq. (9).

Accuracy of these disturbance functions is quite uncertain. Undoubtedly there is a wide variation in accuracy depending on how well the decay factor $\exp(-E\eta)$ was chosen. Certain solutions appear especially poor, those for $A = 0.1$ and 0.3 at $B = 0$ being good examples. In these two cases the two sets of curves represent two decay factors $\exp(-\eta)$ and $\exp(-2\eta)$. The oscillations in the curves near $\eta = 0$ are believed to be imposed by the poor fit of the Galerkin expression; physically, no such oscillation should happen. Probably a still lower value of E would smooth out the curve. Nevertheless, in spite of the considerable disagreement in the shape of these two solutions, the eigenvalue, K, was nearly the same for both. The central group of solutions are expected to have the highest accuracy.

In spite of the uncertain accuracy, the family of plots is informative, for it indicates the approximate height of the disturbance and shows roughly how the height and shape vary with A and B. One of the most notable features is the very strong increase of the v and w components with A; at low values of A there is almost no v and w component. The disturbance plots are useful for throwing light on the energy balance, but no detailed study of such has been made.

Figure 1 shows two vortex patterns for $A = 1$ and $R = 2000$; one is for $B = 0$, the other for $B = 0.1$. These have been computed by the method of isoclines using the v and w components to indicate the flow direction at any point.

The six-term Galerkin solution and the second method produced the following values for K:

$$\left.\begin{aligned} K &= 2.041 \times 10^{-6} \quad \text{By six-term Galerkin} \\ K &= 1.989 \times 10^{-6} \quad \text{By second method} \end{aligned}\right\} A = 1,\ B = 0,\ R = 2000$$

Consequently, the two values of $R_\theta(\theta/r)^{1/2}$ will agree within two per cent. Figure 9 presents the eigenfunctions according to both methods. Because in this case u is by far the strongest component, values of u by the two methods agree rather well. The v solution

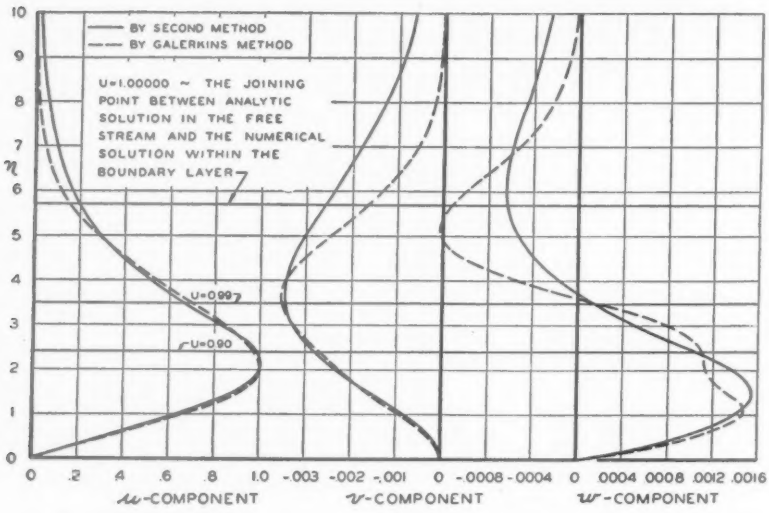

BY SECOND METHOD
BY GALERKINS METHOD
U=1.00000 ~ THE JOINING POINT BETWEEN ANALYTIC SOLUTION IN THE FREE STREAM AND THE NUMERICAL SOLUTION WITHIN THE BOUNDARY LAYER
η
U=0.997
U=0.90
0 .2 .4 .6 .8 1.0 -.003 -.002 -.001 0 -.0008 -.0004 0 .0004 .0008 .0012 .0016
u-COMPONENT
v-COMPONENT
w-COMPONENT

FIG. 9. The disturbance functions according to two methods of calculation.

agrees just as well, but because its magnitude is much less, the percentage accuracy is considerably worse. Since w does not enter directly into the solution and since with $B = 0$, $A = 1$, $w \doteq -dv/d\eta$ slight errors in v will be magnified by w.

Earlier, with regard to solutions for $B = 0$, $A = 0.1$ and 0.3, the odd shape of the u component solutions in Fig. 6 was attributed to the mathematics, for physically such a mode should not exist. The plots of the w component, on Fig. 9, bear out this explanation, for the considerably more accurate solution by the second method does not show the irregularity indicated by the less accurate Galerkin solution.

16. The prediction of transition and agreement with experiment. When a fluid moves past a flat or convex surface, transition of the ordinary type will occur. That is, wavelike disturbances begin and grow, due to Tollmien-Schlichting instability, surface roughness, and turbulence, until the fluctuations are so large that turbulent flow begins to form. When the flow is along a wall having concave curvature, Görtler vortices will develop and grow provided the curvature and boundary layer Reynolds number are sufficiently great. If the curvature is slight, the ordinary type of wave growth will occur, but if it is large, Görtler instability will so predominate that the transition point will be determined entirely by this type of flow.

The actual mechanics of this second type of transition are unknown, since a steady vortex by itself would not necessarily cause transition. Possibly the vortex modifies the boundary layer profile sufficiently to make it unstable to two-dimensional waves. Perhaps, as suggested by Liepmann [5], transition occurs when the apparent shear arising from a disturbance reaches a certain ratio to the basic laminar shear. If so, even though the rate of growth of the vortices is substantially a function of θ only, the actual boundary layer profile can still influence the transition point. For if the apparent shear must reach a certain ratio to the laminar shear, then a more stable profile will require a stronger vortex to maintain the ratio. Certainly it can be said that regardless of the mechanics

of transition, the vortices must attain a certain strength with respect to a characteristic velocity before transition can occur.

In lieu of detailed knowledge of the true development of turbulence, the next best method of determining the effect of concave curvature is to compute, as well as possible, the growth of the vortices, which by some unknown means do cause transition when they become strong enough. That is, since experiment has shown that these vortices have some connection with transition on concave surfaces, the best method of predicting their effect is to follow their growth, as well as possible, to the very point of transition and then to correlate with experiment.

The motion that can be developed by these vortices on plates of variable curvature in a growing boundary layer can be most complex. As conditions change, Fig. 5 shows that of the infinitude of possible vortices, one size will grow strongest, only to lose its ascendancy to one of a different size. All wave-lengths can exist simultaneously. The motion that can be developed may bear little resemblance to that of only a single vortex.

In spite of the possibilities of having a heterogeneous conglomeration of vortices of all sizes and strengths, each and every one obeys the predictions of Fig. 5, provided it is weak. Furthermore, within the limits of the assumption that disturbances are small, the linear nature of the differential equations shows that these disturbances of any wave-length are additive. Therefore, the best means available for following the growth of vortices and their subsequent evolution into turbulent flow appears to be one of evaluating $\int \beta\, dx$ from the point of neutral stability onward. Various wave-lengths as measured by α are tried until the one maximizing $\int \beta\, dx$ is found, for the maximum value of $\int \beta\, dx$ is being sought. By so doing, we determine the factor $\exp \int \beta\, dx$, which indicates the ratio of the strength of the disturbance to its strength at the neutral stability point. What must truly determine transition is the magnitude of these vortices. But $\exp \int \beta\, dx$ is not a magnitude, it is a ratio. To know magnitude one must know the magnitude of the disturbances at the neutral point. Lack of such knowledge constitutes a weak point in the argument for use of $\int \beta\, dx$ as a criterion. Undoubtedly, the magnitude at the neutral point is related to the magnitude of the turbulence in free stream as well as to waves in the boundary layer. Furthermore, the spectrum of the stream turbulence must be considered; certainly the ordinary turbulence value is an inadequate measure, for Liepmann's tests [5] on the effect of turbulence show that the product $(u'/U) \exp \int \beta\, dx$ at transition is not a constant.

Nevertheless, when dealing with wind tunnels of about the same level of turbulence, the available experimental data show $\exp \int \beta\, dx$ at transition to be a better parameter than any used in the past. The results are shown in Fig. 10. The only data available are from the experiments noted on the figure. One set is by Liepmann, obtained from tests on plates of constant curvature [5] in a special tunnel. The velocity along the plate was constant. His investigation covered the case of a growing boundary layer on plates of constant curvature.

A quite different type of test was performed in the small Douglas Wind Tunnel. Transition was determined on the concave side of a 30″ chord, 1/4″ thick, 24*ST* aluminum alloy plate curved to the NACA $a = 1$, $C_{L_{\text{Des}}} = 1$ camber line. The turbulence in this tunnel is low as noted with the data points. In this tunnel, at the test angle of attack of four degrees, the velocity along the concave side is accurately defined by the expression $U/U_\infty = .806(x/c)^{.0545}$; $.01 \leq x/c \leq .70$. In these tests, of course, the radius of curvature varied greatly along the chord.

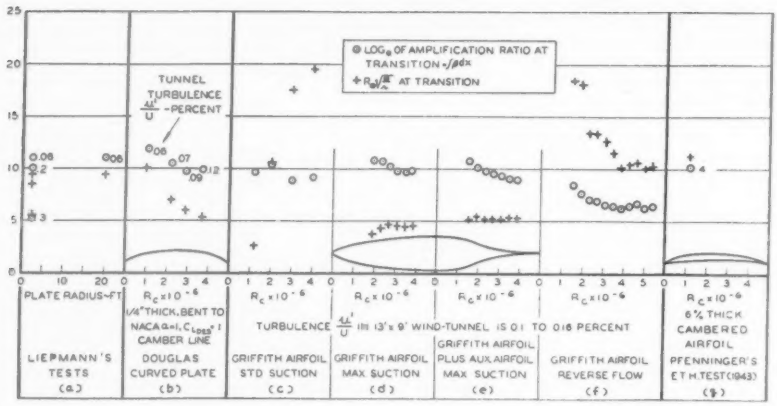

LOG$_e$ OF AMPLIFICATION RATIO AT TRANSITION $=\int \beta dx$
$+ R_\theta\sqrt{\frac{\theta}{r}}$ AT TRANSITION
TUNNEL TURBULENCE $\frac{\sqrt{u'^2}}{U}$ - PERCENT
PLATE RADIUS-FT
$R_c \times 10^{-6}$
1/4" THICK, BENT TO NACA a=1, $C_{L_{DES}}$=1 CAMBER LINE
TURBULENCE $\frac{\sqrt{u'^2}}{U}$ IN 13'x 9' WIND-TUNNEL IS 0.1 TO 0.18 PERCENT
6% THICK CAMBERED AIRFOIL
LIEPMANN'S TESTS (a)
DOUGLAS CURVED PLATE (b)
GRIFFITH AIRFOIL STD SUCTION (c)
GRIFFITH AIRFOIL MAX SUCTION (d)
GRIFFITH AIRFOIL PLUS AUX.AIRFOIL MAX SUCTION (e)
GRIFFITH AIRFOIL REVERSE FLOW (f)
PFENNINGER'S E.T.H. TEST (1943) (g)

FIG. 10. Summary of experimental data on Görtler vortex type of transition.

A third experiment was performed in England while the flow behind the slot on a 16% Griffith airfoil was being examined [12]. Four types of tests were run: one with the normal type of flow and suction, a second with suction so strong as to remove nearly all the boundary layer, a third with the same strong suction but with a velocity distribution modified by means of an auxiliary airfoil, and finally a test with the airfoil facing backwards.

The fourth experiment, conducted by W. Pfenninger at the E.T.H. in Zurich, represents an incidental observation made during tests of a cambered propeller section. The velocity along the concave side was substantially constant back to the observed transition point.

A few comments about each of the tests will now be made. As was to be expected from the theory, Liepmann found that transition occurred at the same value of $R_\theta(\theta/r)^{1/2}$ for either radius of curvature. Both of his tests were for similar flows, that is, constant radius of curvature, constant free stream velocity and a boundary layer that began at the leading edge of the plate. When such conditions apply, the rate parameter $R_\theta(\theta/r)^{1/2}$ also is a measure of the strength of the vortices at transition. Hence, as he verified, $R_\theta(\theta/r)^{1/2}$ is a satisfactory indicator of transition on dimensionally similar flows. In addition, Liepmann's data provide a few valuable clues concerning the effect of tunnel turbulence. However, the rough measure of magnitude of the vortices, $(u'/U)\exp\int \beta\, dx$, is not even approximately constant for the three turbulence levels.

Curvatures of the Douglas curved plate varied from $c/r = 3.5$ to 0.4 to 3.5 from the leading edge to midchord to trailing edge, respectively. To a first approximation, the values of $\int \beta\, dx$ for the most critical wave-length agree with Liepmann's tests. Probably some of the change in $\int \beta\, dx$ with Reynolds number is attributable to changes in the turbulence level noted on the figure, for the trends are in the proper direction.

The next test is a very interesting one, because it represents the case of flow behind a pressure jump on a Griffith airfoil. The general level of the values agrees closely with that of the first two tests. Therefore, the slot itself does not appear to cause a premature transition. Furthermore, the slightly lower level and negative slope of the points can be accounted for partially by the increased turbulence level of this tunnel and by the

effect of speed on turbulence. In the next test, maximum suction was applied in order to remove all the boundary layer so that a fresh boundary layer would develop behind the slot.

On the normal airfoil, the flow from the slot to the trailing edge is accelerating considerably. In order to learn the effect of velocity distribution, the British mounted an auxiliary airfoil nearby to create an essentially constant velocity along this region behind the slot. Figures (10d) and (10e) show that the change had little effect on $\int \beta\, dx$.

Furthermore, these two figures indicate a distinct downward trend of $\int \beta\, dx$ with Reynolds number. In Ref. [12], it is stated that the blower had capacity sufficient to remove the entire boundary layer only at the lower Reynolds numbers. At the maximum Reynolds number only about three-fourths was being removed. Any allowance for such initially thicker boundary layer aft of the slot would appreciably increase $\int \beta\, dx$ at the transition point. Some crude calculations indicate $\int \beta\, dx$ would increase by the order of 0.5 if allowance were made for the thicker boundary layer.

The final Griffith airfoil test represents a reverse flow experiment on the same airfoil. For some unknown reason, the general level of $\int \beta\, dx$ is appreciably less than that for the five other tests. Possibly in such a peculiar test, the boundary layer received outside disturbances. Certainly the flow on the opposite side of the airfoil was separated.

The parameters $R_\theta(\theta/r)^{1/2}$ or a weighted mean of $R_\theta(\theta/r)^{1/2}$ to account for variable curvature have been suggested in the past as indicators of transition. Figure 10 shows that $R_\theta(\theta/r)^{1/2}$ is entirely inadequate; values vary by nearly a tenfold factor. In one experiment, they reduce as a function of Reynolds number; in another, the values increase depending on the distribution of curvature, boundary layer thickness, and boundary layer Reynolds number. Moreover, in several cases, the value of $R_\theta(\theta/r)^{1/2}$ at transition was much lower than values encountered upstream prior to transition where the curvature happened to be greater.

Any weighted mean value of $R_\theta(\theta/r)^{1/2}$ likewise seems doubtful as a valid indicator of transition. If the flow passes a short depression, the curvature may be so high that any type of mean value of $R_\theta(\theta/r)^{1/2}$ over this region would be very large. Yet, the length may be so low that transition will fail to occur. Length of the region should be included to obtain a proper weighting.

Evidently at first, when the vortices are weak and turbulence is low, the flow along a concave wall can be described physically by the methods of this report. In particular the strength grows with distance just as does $\int \beta\, dx$. However, as the vortices become strong, the mathematics no longer describes the motion. But, in spite of a divergence of the mathematical description and physical reality, computation of $\int \beta\, dx$ at transition appears to be sufficiently close to the actual events to provide a usuable indicator of the point of transition.

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AXIAL DISPLACEMENT DISLOCATIONS FOR THE HOLLOW CONE AND THE HOLLOW SPHERE*

BY

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Abstract. Simple solutions in closed form are given for the dislocational states of stress in a hollow cone (Fig. 3) or a hollow sphere (Fig. 5) induced by making an axial cut, and imposing a rigid-body displacement of one face of the cut relative to the other, in the axial direction. The problems are solved by adaptations of the Saint-Venant torsion theory, and of J. H. Michell's theory of torsion of non-uniform shafts.

1. Introduction. A classification of elastic dislocations in a hollow cylinder given by V. Volterra [1] in 1907 is shown in Fig. 1, a dislocation being a state of strain induced

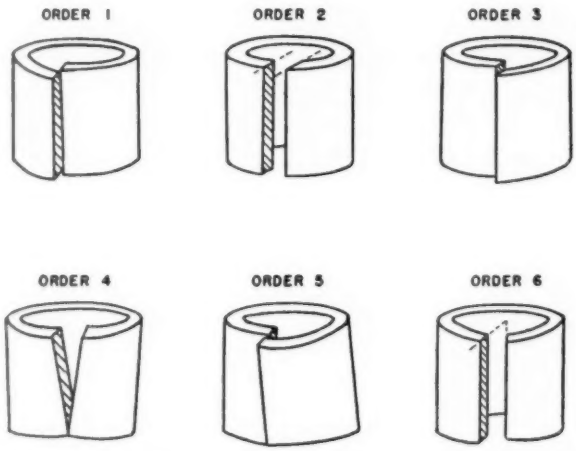

FIG. 1. Volterra's classification of the six dislocations of the hollow cylinder.

by making a cut, imposing relative displacement on the faces, and resealing. The dislocational strain is by definition continuous in the resulting "self-strained" cylinder, at the site of the cut as well as elsewhere, and by a fundamental theorem this requires that the imposed relative displacement be a rigid-body displacement or rotation. We obtain here solutions for the dislocation of order 3 (the relative displacement is an axial translation) in a hollow cone (Fig. 3) and in a hollow sphere (Fig. 5), in simple closed form.

These solutions are derived from the thick-walled cylinder with a dislocation of order 3, by cutting the hollow cone or the hollow sphere out of it, and finding the effects of removing the tractions on the curved surfaces.

As for the cylinder, the dislocation may be conveniently expressed by an adaptation

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of the Saint-Venant torsion theory. In terms of Prandtl's stress function ϕ the equations** of this theory are

$$\tau_{zx} = \frac{\partial \phi}{\partial y}, \qquad \tau_{yz} = -\frac{\partial \phi}{\partial x}, \qquad \nabla^2 \phi = -2G\Theta, \tag{1}$$

ϕ = constant on the boundary of the section, and the warping displacement w can be evaluated from the stress-strain relations

$$\frac{\partial \phi}{\partial y} = G\left(\frac{\partial w}{\partial x} - \Theta y\right), \qquad -\frac{\partial \phi}{\partial x} = G\left(\frac{\partial w}{\partial y} + \Theta x\right). \tag{2}$$

The adaptation consists in making the twist Θ zero. In the torsion problem itself this corresponds merely to the unstressed state. The system of equations nevertheless remains capable of expressing certain non-torsional states. Equations (2) become Cauchy-Riemann equations showing that (with $z = x + iy$, temporarily)

$$Gw + i\phi = f(z). \tag{3}$$

A discontinuity in w of the type required by the dislocation of order 3 is introduced by choosing

$$f(z) = -iA \log z = A\theta - iA \log r, \tag{4}$$

r and θ being polar co-ordinates in the cross-section. Then with A as a real arbitrary constant

$$Gw = A\theta, \qquad \phi = -A \log r \tag{5}$$

and the polar components of shear stress on the section are (z now being the cylindrical co-ordinate in r, θ, z)

$$\tau_{z\theta} = -\frac{\partial \phi}{\partial r} = \frac{A}{r}, \qquad \tau_{zr} = 0. \tag{6}$$

Cylindrical surfaces r = constant are free from stress. The stress is attributable to the discontinuity in w

$$w(r, 2\pi) - w(r, 0) = 2\pi A/G \tag{7}$$

at the faces of an axial cut ($\theta = 0$, $\theta = 2\pi$), but also partly to the torque on the ends ($a < r < b$)

$$\int_a^b \tau_{z\theta} 2\pi r^2 \, dr = \pi(b^2 - a^2)A.$$

This torque can be removed by superimposing ordinary torsion

$$\tau_{z\theta} = Br, \qquad \tau_{zr} = 0 \qquad \text{with} \qquad B = -2A/(a^2 + b^2), \tag{8}$$

leaving stress

$$\tau_{z\theta} = A\left(\frac{1}{r} - \frac{2r}{a^2 + b^2}\right) \tag{9}$$

**The notation is that of Timoshenko and Goodier, *Theory of elasticity*, Ch. 11, 1951, except that Θ is used for unit twist instead of θ.

which may be attributed to the dislocational displacement expressed by (7). This non-zero stress prevails on the ends of course as well as throughout the cylinder, but since it has zero resultant its removal would affect only localized end zones.*

2. The hollow cone. In the dislocated cylinder the cylindrical surfaces r = constant are free from traction, but on other surfaces of revolution $f(r, z) = 0$ the stress (9) implies ringwise shear loading, and the solution of our problems requires removal of this.

The problem presented by ringwise shear loading on a body of revolution can be formulated as an adaptation of J. H. Michell's theory of torsion of non-uniform circular shafts. In that theory the equations of equilibrium are satisfied by taking

$$\tau_{r\theta} = -\frac{1}{r^2}\frac{\partial \psi}{\partial z} \qquad \tau_{z\theta} = \frac{1}{r^2}\frac{\partial \psi}{\partial r}. \tag{10}$$

The remaining four stress components are zero. Compatibility of strain requires

$$\frac{\partial}{\partial r}\left(\frac{1}{r^3}\frac{\partial \psi}{\partial r}\right) + \frac{\partial}{\partial z}\left(\frac{1}{r^3}\frac{\partial \psi}{\partial z}\right) = 0. \tag{11}$$

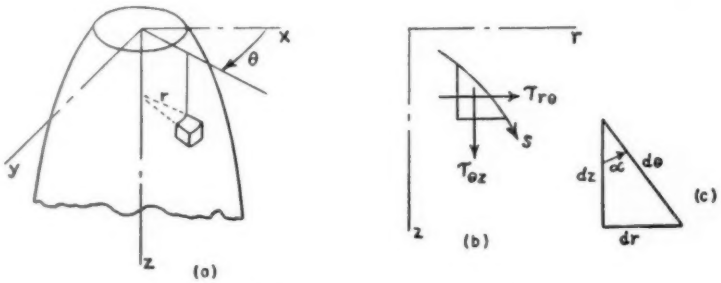

FIG. 2. Torsional stress in a body of revolution.

In Fig. 2(b) the stresses $\tau_{r\theta}$, $\tau_{z\theta}$ are shown on an *axial* section of the solid of revolution in Fig. 2(a), at the boundary (s). The shear stress formed by these two components can be represented as components normal and tangential to the boundary curve. By (10), the normal component $\tau_{\theta n}$ (outward—to the right—in Fig. 2b) is

$$\tau_{\theta n} = \tau_{r\theta}\cos\alpha - \tau_{\theta z}\sin\alpha = -\frac{1}{r^2}\frac{\partial \psi}{\partial s}. \tag{12}$$

In the Michell torsion theory this normal component at the boundary is zero, and ψ is constant, because the boundary surface is free. The adaptation to our problem consists simply in retaining (12) with a non-zero prescribed $\tau_{\theta n}$ as a function of z (or r) on the boundary curve.

*This analysis explains a paradox of thin shell theory. If a thin flat rectangular plate is put under pure shear it becomes a nearly rectangular parallelogram. It can be rolled up into a cylinder, but the joined edges (along a generator) will not quite register. A dislocational axial displacement remains. The shears on the edges remain, and form torques on the ends. It would appear that removal of these torques by simple twisting of the closed cylinder will remove the shear stress altogether, and leave the cylinder dislocated but unstressed. The paradox disappears when variation of stress through the thickness is admitted as in (9).

The hollow cone (Fig. 3a) is first taken as part of a hollow cylinder in the state expressed by (5), (6), and (7). Spherical co-ordinates R, ω, θ, indicated in Fig. 3 are used

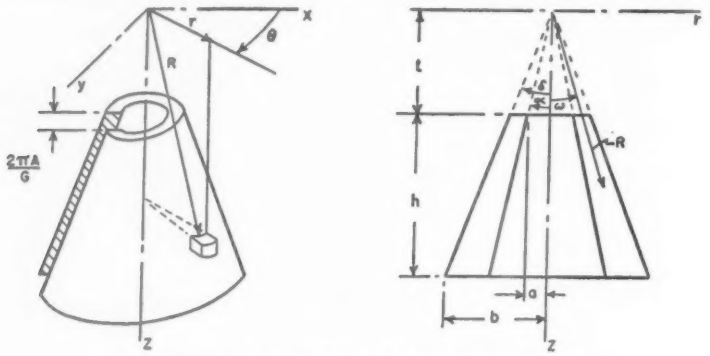

FIG. 3. The hollow cone with a dislocation of order 3.

for convenience instead of the cylindrical co-ordinates r, θ, z. An element at a conical surface (Fig. 4) has the cylindrical shear component $\tau_{z\theta}$ of (6) as the only non-zero stress in cylindrical co-ordinates, and in spherical co-ordinates this gives rise to two components $\tau_{\theta\omega}$ and $\tau_{\theta R}$, normal and tangential to the conical surface. The normal component $\tau_{\theta\omega}$ is shown in Fig. 4, together with its complementary shear $\tau_{\omega\theta}$ *on* the

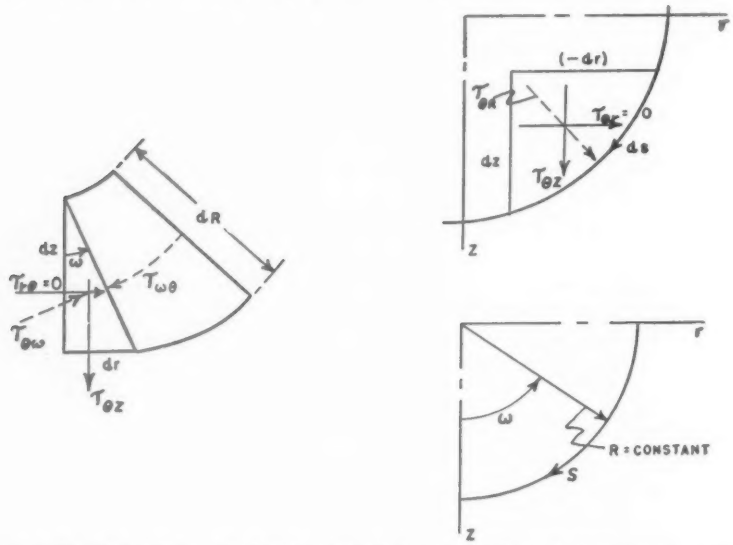

FIG. 4. Transition from cylindrical to spherical co-ordinates and stress components.

conical surface—the ringwise shear which must be removed from the conical boundary surfaces $\omega = \alpha$ and $\omega = \delta$. We have, from Fig. 4 and Eq. 6

$$\tau_{\theta\omega} = \tau_{\omega\theta} = -\tau_{\theta z} \sin \omega = -\frac{A}{r} \sin \omega = -\frac{A}{R}. \tag{13}$$

In spherical co-ordinates Michell's equations (10) yield

$$\tau_{\omega\theta} = -\frac{1}{r^2}\frac{\partial\psi}{\partial R} = -\frac{1}{R^2\sin^2\omega}\frac{\partial\psi}{\partial R} \tag{14}$$

and the differential equation (11) for ψ becomes, writing $\mu = \cos\omega$

$$\frac{\partial}{\partial R}\frac{1}{R^2}\frac{1}{(1-\mu^2)^2}\frac{\partial\psi}{\partial R} + \frac{\partial}{\partial\mu}\frac{1}{R^4}\frac{1}{(1-\mu^2)}\frac{\partial\psi}{\partial\mu} = 0. \tag{15}$$

In view of (14) and (13) we require a solution of this having a derivative $\partial\psi/\partial R$ which behaves like R on the surfaces $\omega = \alpha$ and $\omega = \delta$. The required form [2] is

$$\psi = R^2[B(1+\mu^2) + C\mu], \tag{16}$$

with B and C arbitrary constants.

Combining the state represented by (16) with the state represented by (5), (6), and (7) we find that the stress component $\tau_{\omega\theta}$ is given by

$$\tau_{\omega\theta} = -\frac{A}{R} - \frac{2B}{R}(1 + 2\cot^2\omega) - \frac{2C}{R}\frac{\cot\omega}{\sin\omega}.$$

For freedom of the conical surfaces this should vanish when $\omega = \alpha$ and when $\omega = \delta$, and therefore B and C are determined as

$$\begin{aligned} B &= -\tfrac{1}{2}AH(\sin^2\alpha\cos\delta - \sin^2\delta\cos\alpha), \\ C &= -\tfrac{1}{2}AH[(1+\cos^2\alpha)\sin^2\delta - (1+\cos^2\delta)\sin^2\alpha], \end{aligned} \tag{17}$$

where $H = [(1+\cos^2\alpha)\cos\delta - (1+\cos^2\delta)\cos\alpha]^{-1}$. The constant A is determined from (7) by the prescribed displacement discontinuity.

The state of stress so found is conveniently stated in cylindrical co-ordinates as

$$\begin{aligned} \tau_{\theta z} &= (A + 2B)r^{-1} + Czr^{-1}(r^2+z^2)^{-1/2}, \\ \tau_{r\theta} &= -4Bzr^{-2} - Cr^{-2}(r^2+2z^2)(r^2+z^2)^{-1/2}, \\ \sigma_r &= \sigma_\theta = \sigma_z = \tau_{zr} = 0. \end{aligned} \tag{18}$$

The condition of zero torque on the ends has not been imposed. Evaluation of the torque from $\tau_{\theta z}$ in (18) shows it to be zero however, and therefore (18) represents the desired solution for the dislocation of order 3 in the hollow cone. Shear corresponding to $\tau_{\theta z}$ remains as a self-equilibrating distribution on each end, and its removal would alter the stress near the ends.

3. The hollow sphere. The hollow sphere, or spherical frustum, shown in Fig. 5, is taken first as part of a hollow cylinder in the state expressed by (5), (6), and (7). The shear stress $\tau_{\theta R}$ or $\tau_{R\theta}$, in spherical co-ordinates R, ω, θ, is (Fig. 4)

$$\tau_{R\theta} = -\tau_{\theta z}\frac{dr}{ds} = -\frac{A}{r}\frac{dr}{ds}.$$

For a spherical surface of radius R, $dr/ds = -z/R$. On the boundary surfaces $R = a$, $R = b$ we have therefore

$$(\tau_{R\theta})_{R=a} = \frac{A}{a}\left(\frac{z}{r}\right)_{R=a}; \qquad (\tau_{r\theta})_{R=b} = \frac{A}{b}\left(\frac{z}{r}\right)_{R=b}. \tag{19}$$

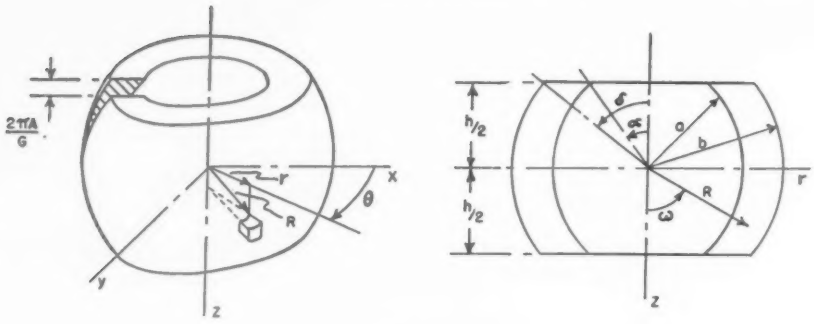

FIG. 5.

To remove these we introduce a Michell function ψ according to (11) and (10). In spherical co-ordinates this leads, by (12), to

$$\tau_{R\theta} = -\frac{1}{r^2}\left(\frac{\partial \psi}{\partial s}\right)_R$$

and so, in view of (19), the boundary conditions on ψ are

$$\left(\frac{\partial \psi}{\partial s}\right)_{R=a} = Aa \sin \omega \cos \omega, \qquad \left(\frac{\partial \psi}{\partial s}\right)_{R=b} = Ab \sin \omega \cos \omega. \tag{20}$$

On a surface R = constant we have $ds = -R\, d\omega$, and with $\mu = \cos \omega$ we may write for this surface

$$\frac{\partial \psi}{\partial s} = -\frac{1}{R}\frac{\partial \psi}{\partial \omega} = -\frac{1}{R}\frac{\partial \psi}{\partial \mu}\frac{d\mu}{d\omega} = \frac{1}{R}\sin \omega \frac{\partial \psi}{\partial \mu}.$$

Introducing this in (20) yields

$$\left(\frac{\partial \psi}{\partial \mu}\right)_{R=a} = Aa^2\mu, \qquad \left(\frac{\partial \psi}{\partial \mu}\right)_{R=b} = Ab^2\mu. \tag{21}$$

The solution of (15) appropriate to these conditions is the B solution from (16),

$$\psi = BR^2(1 + \mu^2)$$

and both conditions (21) are satisfied if $B = A/2$.

Superposition of the stress represented by this function, and the stress (6), gives the following state of stress in cylindrical co-ordinates

$$\begin{aligned} \tau_{\theta z} &= 2A/r, \qquad \tau_{r\theta} = -2Az/r, \\ \sigma_r &= \sigma_\theta = \sigma_z = \tau_{zr} = 0. \end{aligned} \tag{22}$$

The torque on the ends of the hollow spherical frustum (Fig. 5) is

$$T = \int_{a \sin \alpha}^{b \sin \delta} \tau_\theta 2\pi r^2\, dr = 2\pi A(b^2 \sin^2 \delta - a^2 \sin^2 \alpha). \tag{23}$$

It may be removed by means of the ψ-function corresponding to torsion of a sphere

$$\psi = FR^3 \qquad (F \text{ a constant}).$$

On a spherical surface R = constant this function is constant, and the surface is consequently free. The stress and torque are (reverting to cylindrical co-ordinates)

$$\tau_{\theta z} = 3Fr^{-1}(r^2 + z^2)^{1/2}, \qquad \tau_{r\theta} = -3Fzr^{-2}(r^2 + z^2)^{1/2},$$
$$T = \int_{a \sin \alpha}^{b \sin \delta} \tau_{\theta z} 2\pi r^2 \, dr = 2\pi F(b^3 - a^3). \tag{24}$$

Combining this with (22), (23), vanishing of the total torque requires $F = -A(b^2 \sin^2 \delta - a^2 \sin^2 \alpha)(b^3 - a^3)^{-1}$, and we find as the solution for the dislocation of order 3 for the hollow spherical frustum, corresponding to the displacement discontinuity (7),

$$\tau_{\theta z} = A[2r^{-1} - 3r^{-1}(r^2 + z^2)^{1/2}(b^2 \sin^2 \delta - a^2 \sin^2 \alpha)(b^3 - a^3)^{-1}],$$
$$\tau_{r\theta} = A[-2zr^{-2} + 3zr^{-2}(r^2 + z^2)^{1/2}(b^2 \sin^2 \delta - a^2 \sin^2 \alpha)(b^3 - a^3)^{-1}], \tag{25}$$
$$\sigma_r = \sigma_\theta = \sigma_z = \tau_{zr} = 0.$$

Again a self-equilibrating distribution of $\tau_{\theta z}$ remains on the ends.

All these dislocations, for cylinder, cone, and sphere, have singularities in the stress on the axis $r = 0$, necessitating the restriction to the hollow cylinder and the frustum of the cone or sphere.

Ghosh [3] has given the dislocations of orders 2 and 6 (Fig. 1) for the hollow sphere. The result (25) adds a third to this set.

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1. V. Volterra, *Sur l'equilibre des corps élastiques multiplement connexés*, Ann. Sci. de l'Ecole Normale Supérieure, Ser. 3, **24**, 401 (1907)
2. J. C. Wilhoit, Jr., *An addition to Poritsky's solution of a differential equation of torsion*, Quart. Appl. Math. **11**, 499 (1954)
3. S. Ghosh, *On some many-valued solutions of the equations of elastic equilibrium in polar co-ordinates*, Z. angew. Math. u. Mech. **12**, 188 (1932)

BOOK REVIEWS

Table of binomial coefficients. Royal Society Mathematical Tables, Volume 3. Under the Editorship of J. C. P. Miller. Cambridge University Press, 1954. viii + 162 pp. $6.50.

This is the most extensive table of binomial coefficients yet published. The coefficients $\binom{n}{r}$ are given exactly for all integer $r(r \leq n/2)$ and n up to $n = 200$. In addition the table includes larger values of n up to $n = 5000$ but for a limited range of r, to $r = 12$ for $200 < n < 500$, to $r = 11$ for $500 < n < 1000$, to $r = 5$ for $1000 < n < 2000$ and to $r = 3$ for $2000 < n < 5000$.

The tables are well arranged especially considering that some coefficients require only one digit whereas some others require almost 60 digits.

The largest previous table of binomial coefficients (by Peters and Stein 1922) was complete only to $n = 60$.

G. Newell

Linear transient analysis. Volume I. By Ernst Weber. John Wiley & Sons, Inc., New York, and Chapman & Hall, Ltd., London, 1954. xiv + 348 pp. $7.50.

At a time when the use and misuse of the Laplace transform is enjoying great popularity, this careful and basic study and critical comparison of a variety of methods for the analysis of transient phenomena is particularly welcome. After a critical examination of the circuit concept as derived from the electromagnetic field relations in the opening chapter, the formulation and classical solution of the integro-differential equations of linear networks are reviewed. One chapter is devoted to Heaviside's operational calculus, with modifications given it by H. Jeffreys which allow a rigorous and systematic treatment. The longest chapter deals with the Laplace transform and its use in electric circuit problems. The expression for the inverse transform is here obtained from Cauchy's integral formula which is developed in an 11-page appendix. The last chapter discusses the frequency spectrum concept, Fourier integrals, and the Fourier transform method. A very good discussion of electrical, mechanical, and thermal analogues is given in one chapter. Six appendices include some of the mathematical background needed in the text as well as a general bibliography. The text is exceptionally well documented by means of footnote references and other reference lists.

This book is intended as a text for seniors or first-year graduate students in electrical engineering. The applications in this volume are essentially limited to two-terminal, lumped-parameter networks; two terminal-pair networks, filters, and transmission lines are to be treated in a second volume. The broad coverage of methods of analysis, the extensive use of examples to introduce or illustrate the methods, the emphasis on physical concepts, and the clear exposition all contribute to make this an excellent textbook.

R. J. Schwarz

Limit distributions for sums of independent random variables. By B. V. Gnedenko and A. N. Kolmogorov. Translated by K. L. Chung. Addison-Wesley Publishing Co., Inc., Cambridge 42, Massachusetts, 1954. ix + 264 pp. $7.50.

This translation of a recent Russian book will be of most value to advanced students and specialists in the mathematics of probability theory. Much of the book covers material that has previously appeared only in periodicals, many in Russian. Although the first few chapters give a general review of the fundamentals of probability theory (in terms of measure theory), most of the book consists of a rather exhaustive and rigorous study of the restricted field described by the title.

G. Newell

(Continued on p. 340)

THE MOTION OF A VISCOUS FLUID CONDUCTING HEAT*

BY

J. L. SYNGE

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1. Introduction. The purpose of this paper is to draw attention to the comparatively simple equations governing the behaviour of a viscous compressible fluid which conducts heat. In these equations (following the suggestion of a referee who read the paper in its original and slightly different form) I use specific entropy and specific volume as basic thermodynamic variables. In this respect the argument differs from that of Eckart[1], who derived the equations using pressure and specific volume. I do not think that the difference is important, and I would not seek to publish my derivation of the equations were it not for the intolerable barrier which separates hydrodynamics from thermodynamics—a barrier which can be worn down only by repetition of the fact that there exist clearly formulated equations governing the behaviour of a viscous fluid conducting heat. Believing that a system of equations takes on reality only when one uses them, I have added some work on exponential solutions of the equations, linearised for the case of small disturbances from equilibrium.

Anyone who consults the compendious survey by Truesdell[2] will realise that the history of the mechanics of continua is no simple thing. But it seems that J. W. Gibbs (1839-1903) pointed the way towards the equations in question, and might have derived them eighty years ago if he had been interested in hydrodynamics. Whether the equations emerged in the interval and were lost to sight I do not know; the first derivation that I know of is that of Goldstein[3], who avoided the use of entropy and thus made the work much more difficult to follow than Eckart's.

This avoidance of entropy may well be the key to the question: Why have the mechanics of continua and thermodynamics kept so far apart? For we do not find the word *entropy* in the indexes of the standard works of Lamb[4], Love[5], and Goldstein (op. cit.); nor do we find it in the index of Dryden, Murnaghan and Bateman[6]; Milne-Thomson[7] introduces entropy only for a perfect gas.

There are curious time-lags in science. Maxwell's equations took some fifty years to gain full control of optics. The ideas of Gibbs have had an even greater delay in taking their due place in hydrodynamics. I believe that the cause is the same in both cases—a horror of abstraction. My guess is that the still prevalent entrophoby (to coin a word) springs from an unwritten law that mechanics should deal only with concepts directly related to ordinary experience (pressure, density, velocity, temperature, energy); in a macroscopic theory entropy must remain something rather mysterious, its physical interpretation (as a measure of disorder) coming to light only in the micro-

*Received August 13, 1954. Revised version received November 26, 1954.

[1]C. Eckart, Phys. Rev. (2) **58**, 267-269 (1940).

[2]C. Truesdell, J. of Rational Mechanics and Analysis **1**, 125-300 (1952).

[3]S. Goldstein, *Modern developments in fluid dynamics*, Oxford, Clarendon Press, 1938, vol. 2, pp. 601-606.

[4]Sir H. Lamb, *Hydrodynamics*, Cambridge University Press, 1932.

[5]A. E. H. Love, *The mathematical theory of elasticity*, Cambridge University Press, 1934.

[6]Bulletin of the National Research Council, No. 84, 1932.

[7]L. M. Milne-Thomson, *Theoretical hydrodynamics*, Macmillan, London, 1949.

scopic view (matter resolved into molecules), a view quite out of keeping with the spirit of hydrodynamics, as ordinarily understood. However commendable this commonsense attitude may be, the price we have to pay for it in intellectual confusion is too high, and my own feeling is that thermo-hydrodynamics takes on a clear mathematical form only when entropy is given a leading role. Just what this ghostly visitor amounts to must come out *a posteriori* (how does it behave in the mathematical theory?) and not *a priori* (how is it connected with our ordinary sensory perceptions?).

Entrophoby must be fought until such time as entropy takes the place it deserves in standard texts on hydrodynamics. One must keep on repeating that the equations are there and that they are understandable. Eckart's paper is only three pages long and is easily overlooked, as it apparently was by Courant and Friedrichs[8], who, though otherwise commendably entrophile, refer the reader to Goldstein's derivation of the equations.

It is unfortunate that in a recent book de Groot[9] has missed an opportunity to pick these equations out from surrounding thermodynamical complexities and make them readily available to mathematicians. The equations must surely be on pp. 95, 96 of his book, if we reduce the number of components to one. But I cannot find them in their entirety, since the Gibbs equation [(3.2) of the present paper] appears as an *ordinary* differential equation in de Groot's work, whereas to me (and surely it was so to Gibbs) it is a *total* differential equation in the space of the thermodynamic variables, a very different thing mathematically.

Since the purpose of this paper calls for emphasis on the simplest form of equations available, I have taken the case where there is only one coefficient of viscosity. But one can easily pass to the more general case of two coefficients, and so obtain a mathematical model closer to nature[10].

2. Notation. We are concerned with Newtonian mechanics in which mass is conserved, and specific quantities will be measured per unit mass. There is something to be said for an alternative plan in which specific quantities are measured per unit standard volume (i.e. volume at standard temperature and pressure), and this is the better plan in relativity, since mass (as a form of energy) is no longer conserved. But in Newtonian mechanics the two plans are equivalent and it seems a little simpler to measure per unit mass, which is the usual way.

Latin suffixes take the values 1, 2, 3, with the summation convention for repeated suffixes; x_i are rectangular Cartesian coordinates in any Newtonian frame of reference (i.e. an unaccelerated frame), and partial derivatives with respect to the coordinates are indicated by a comma ($f_{,i} = \partial f/\partial x_i$).

The following notation will be used (*specific* means *per unit mass*): u_i = velocity, p = pressure, T = absolute temperature, ρ = density, $\sigma = 1/\rho$ = specific volume, S = specific entropy, U = specific internal energy, μ = viscosity, κ = thermal conductivity, $E_{ij} = E_{ji}$ = stress tensor, $e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$ = rate of deformation tensor, $d\tau$ = element of volume of a region of the fluid bounded by a surface B always composed

[8]R. Courant and K. Friedrichs, *Supersonic flow and shock waves*, Interscience Publishers Inc., New York, 1948, p. 134.

[9]S. R. de Groot, *Thermodynamics of irreversible processes*, North-Holland Publishing Co., Amsterdam, 1951.

[10]See C. Truesdell, J. of Rational Mechanics and Analysis 2, 643-741 (1953); and L. Rosenhead *et al.*, *A discussion on the first and second viscosities of fluids*, Proc. Roy. Soc. London **A226**, 1-69 (1954).

of the same particles, dB = element of B, n_i = unit outward normal to B, X_i = specific body force, h_i = heat-flow vector.

3. The entropy equation. The quantities S, T, U, p, and σ (equivalently ρ) are *thermodynamic variables*. We shall select S and σ as *basic* thermodynamic variables; the values of S and σ at any event (x, t) determine the *state* of the fluid at that event. We assume an *equation of state*

$$U = f(S, \sigma), \tag{3.1}$$

and further assume that the other two thermodynamic variables are also functions of S and σ consistent with the *entropy equation*

$$T\,dS = dU + p\,d\sigma, \tag{3.2}$$

which is to hold for arbitrary infinitesimals dS, $d\sigma$, and thus is equivalent to

$$T = U_S\,, \qquad p = -U_\sigma\,, \tag{3.3}$$

the subscripts denoting partial derivatives with respect to S and σ.

The quantities μ, κ are assumed to be given *positive* functions of S and σ.

4. The conservation of mass, the equation of momentum, and the first law of thermodynamics. The conservation of mass gives

$$\frac{d\rho}{dt} + \rho u_{k,k} = 0, \tag{4.1}$$

where for any function $F(x, t)$

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + F_{,k}u_k\,, \tag{4.2}$$

the derivative following the fluid.

Following a portion of the fluid which consists always of the same particles, the law of momentum gives

$$\frac{d}{dt}\int \rho u_i\,d\tau = \int \rho X_i\,d\tau + \int E_{ij}n_j\,dB, \tag{4.3}$$

and hence, since $\rho d\tau$ is conserved,

$$\rho\frac{du_i}{dt} = \rho X_i + E_{ij,j}\,. \tag{4.4}$$

Again following the fluid, the first law of thermodynamics gives

$$\frac{d}{dt}\int \tfrac{1}{2}\rho u_i u_i\,d\tau + \frac{d}{dt}\int \rho U\,d\tau = \int \rho X_i u_i\,d\tau + \int E_{ij}n_j u_i\,dB - \int h_i n_i\,dB, \tag{4.5}$$

and this, when the surface integrals are changed to volume integrals and (4.4) is used, leads to

$$\rho\frac{dU}{dt} = E_{ij}e_{ij} - h_{i,i}\,. \tag{4.6}$$

5. Stress-deformation equations and thermal conduction. In listing the notation in Sec. 2 the pressure p and stress E_{ij} were not connected. We now assume

$$3p = -E_{kk}\,. \tag{5.1}$$

As stress-deformation equations we assume

$$E_{ij} = -p\delta_{ij} + 2\mu(e_{ij} - \tfrac{1}{3}\delta_{ij}e_{kk}), \tag{5.2}$$

where δ_{ij} is the Kronecker delta.

In view of (5.1) there are five equations in (5.2), not six, since the contraction $j = i$ gives an identity. On the other hand (5.2) implies (5.1).

As law of thermal conduction we assume

$$h_i = -\kappa T_{,i} . \tag{5.3}$$

6. Muster of equations. Substituting from (5.2) and (5.3) in (4.1), (4.4), and (4.6) we get the following equations:

$$\frac{d\rho}{dt} + \rho e_{kk} = 0, \tag{6.1}$$

$$\rho \frac{du_i}{dt} = \rho X_i - p_{,i} + 2[\mu(e_{ij} - \tfrac{1}{3}\delta_{ij}e_{kk})]_{,j} , \tag{6.2}$$

$$\rho \frac{dU}{dt} = -pe_{kk} + 2\mu(e_{ij}e_{ij} - \tfrac{1}{3}e_{kk}^2) + (\kappa T_{,i})_{,i} . \tag{6.3}$$

In these equations $e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$, and we recognise here five equations for the five unknowns u_i, S, σ, for $\rho = 1/\sigma$ and by Sec. 3 U, T, and p are known functions of S and σ. *These are the desired equations governing the behaviour of a viscous fluid which conducts heat.* They give a determinate problem in the sense that the number of equations equals the number of unknowns; we shall not attempt to discuss what boundary conditions are consistent with determinacy of the solution.

Equation (6.3) may be written a little differently. By (3.2), following the fluid, we have

$$T\frac{dS}{dt} = \frac{dU}{dt} + p\frac{d\sigma}{dt} = \frac{dU}{dt} - \frac{p}{\rho^2}\frac{d\rho}{dt} = \frac{dU}{dt} + \frac{p}{\rho}e_{kk} , \tag{6.4}$$

and so (6.3) is equivalent to

$$\rho T\frac{dS}{dt} = 2\mu(e_{ij}e_{ij} - \tfrac{1}{3}e_{kk}^2) + (\kappa T_{,i})_{,i} . \tag{6.5}$$

7. Small disturbances from equilibrium. We suppose body force absent ($X_i = 0$) and consider a state of equilibrium in which $u_i = 0$ and the thermodynamic variables have constant values which we shall denote by $S^{(0)}$, $\sigma^{(0)}$, etc. For a small disturbance we have u_i small and

$$S = S^{(0)} + S^{(1)}, \qquad \sigma = \sigma^{(0)} + \sigma^{(1)}, \qquad \text{etc.}, \tag{7.1}$$

where the quantities marked (1) are small. We linearise the non-linear equations (6.1), (6.2), (6.5) by neglecting terms quadratic in small quantities. Thus we get

$$\frac{\partial\rho^{(1)}}{\partial t} + \rho^{(0)}u_{k,k} = 0, \tag{7.2}$$

$$\rho^{(0)}\frac{\partial u_i}{\partial t} = -p_{,i}^{(1)} + \mu^{(0)}(\Delta u_i + \tfrac{1}{3}u_{k,ki}), \tag{7.3}$$

$$\rho^{(0)}T^{(0)}\frac{\partial S^{(1)}}{\partial t} = \kappa^{(0)}\Delta T^{(1)}, \tag{7.4}$$

where Δ is the Laplacian operator.

By (3.3) we have in equilibrium

$$T^{(0)} = U_S^{(0)}, \qquad p^{(0)} = -U_\sigma^{(0)}. \tag{7.5}$$

For the disturbed state

$$T^{(1)} = U_{SS}^{(0)} S^{(1)} + U_{S\sigma}^{(0)} \sigma^{(1)}, \qquad p^{(1)} = -U_{\sigma S}^{(0)} S^{(1)} - U_{\sigma\sigma}^{(0)} \sigma^{(1)}, \tag{7.6}$$

where the coefficients are known constants. Since $\rho = 1/\sigma$, we have

$$\rho^{(1)} = -\rho^{(0)2} \sigma^{(1)}. \tag{7.7}$$

Substitution in (7.2), (7.3), (7.4) gives the following five linear equations with constant coefficients for the determination of u_i, $S^{(1)}$, $\sigma^{(1)}$:

$$\left.\begin{aligned} &\frac{\partial \sigma^{(1)}}{\partial t} - \sigma^{(0)} u_{k,k} = 0, \\ &\rho^{(0)} \frac{\partial u_i}{\partial t} - U_{\sigma S}^{(0)} S_{,i}^{(1)} - U_{\sigma\sigma}^{(0)} \sigma_{,i}^{(1)} - \mu^{(0)} (\Delta u_i + \tfrac{1}{3} u_{k,ki}) = 0, \\ &\rho^{(0)} T^{(0)} \frac{\partial S^{(1)}}{\partial t} - \kappa^{(0)} (U_{SS}^{(0)} \Delta S^{(1)} + U_{S\sigma}^{(0)} \Delta \sigma^{(1)}) = 0. \end{aligned}\right\} \tag{7.8}$$

8. Elementary solutions. Let us now investigate elementary solutions of (7.8) of the form

$$\left.\begin{aligned} u_i = u_i^* \exp(a_i x_i + bt), \qquad S^{(1)} = S^* \exp(a_i x_i + bt), \\ \sigma^{(1)} = \sigma^* \exp(a_i x_i + bt), \end{aligned}\right\} \tag{8.1}$$

where u_i^*, S^*, σ^*, a_i, b, are nine constants, in general complex. This is a general approach to the special problems considered by Lamb (op. cit., pp. 645-657).

Splitting a_i and b into real and imaginary parts,

$$a_i = a_i' + i a_i'', \qquad b = b' + i b'', \tag{8.2}$$

we have

$$\sigma^{(1)} = \sigma^* \exp(a_i' x_i + b't) \exp i(a_i'' x_i + b'' t), \tag{8.3}$$

and similar expressions for u_i and $S^{(1)}$; the physical disturbances are of course the real parts of these complex expressions. A realisation of the meaning of expressions like (8.3) is helped by considering the equations

$$a_i' x_i + b' t = \text{constant}, \tag{8.4}$$

$$a_i'' x_i + b'' t = \text{constant}, \tag{8.5}$$

each of which represents a family of moving planes (or waves), or equivalently a family of 3-flats in Newtonian space-time. It is clear that (8.4) are *amplitude waves*, the amplitude of the disturbance being propagated unchanged in the direction of the vector a_i' with velocity

$$V_i' = -a_i' b' / (a_k' a_k'), \tag{8.6}$$

and (8.5) are *phase waves*, the phase being propagated unchanged with velocity

$$V_i'' = -a_i'' b'' / (a_k'' a_k''). \tag{8.7}$$

The amplitude waves have neither wave length nor period; the wave length and period of the phase waves are

$$\lambda = 2\pi(a_i' a_i')^{-1/2}, \qquad \tau = 2\pi \mid b'' \mid^{-1}. \tag{8.8}$$

The complex vector a_i is closely related to heat flow. For the complex heat flow vector is by (7.6), to the first order,

$$h_i = -\kappa^{(0)} T^{(1)}_{,i} = -\kappa^{(0)}[U^{(0)}_{SS} S^* + U^{(0)}_{S\sigma}\sigma^*]a_i \exp(a_i x_i + bt), \tag{8.9}$$

and so the real heat flow vector is parallel to the planes which are parallel to the two real vectors a_i' , a_i'' .

We now substitute from (8.1) in (7.8), but we shall simplify the notation by dropping the label (0), so that *henceforth* ρ, U_{SS} , etc. *refer to the equilibrium state*. We get the following five equations:

$$\left.\begin{aligned} \rho\sigma^* b - u_k^* a_k &= 0, \\ \rho u_i^* b - a_i(U_{\sigma S} S^* + U_{\sigma\sigma}\sigma^*) - \mu(u_i^* a_k a_k + \tfrac{1}{3} u_k^* a_k a_i) &= 0, \\ \rho T S^* b - \kappa a_k a_k (U_{SS} S^* + U_{S\sigma}\sigma^*) &= 0. \end{aligned}\right\} \tag{8.10}$$

It is convenient to subtract from the second line the first multiplied by $\frac{1}{3}\mu a_i$, so that our set of equations may be written

$$\left.\begin{aligned} \rho b\sigma^* - u_k^* a_k &= 0, \\ a_i[U_{\sigma S} S^* + (U_{\sigma\sigma} + \tfrac{1}{3}\mu\rho b)\sigma^*] + u_i^*[\mu a_k a_k - \rho b] &= 0, \\ \rho T b S^* - \kappa a_k a_k (U_{SS} S^* + U_{S\sigma}\sigma^*) &= 0. \end{aligned}\right\} \tag{8.11}$$

Any choice of the nine complex constants u_i^* , S^*, σ^*, a_i , b satisfying these five equations gives us an elementary disturbance (8.1). To investigate the solutions of (8.11) an obvious plan is to eliminate the first five quantities, obtaining a 5×5 determinantal equation connecting a_i and b, but the algebra may be reduced by using special axes.

9. Reduction of the algebraic problem. If a_i were a real vector, we could make $a_2 = a_3 = 0$ by choosing the x_1-axis in the direction of the vector a_i . Since a_i is in general complex, we cannot do this, but we can choose the x_3-axis perpendicular to the two real vectors a_i' , a_i'' and so make

$$a_3 = 0. \tag{9.1}$$

Then, with Greek suffixes for the range 1, 2, our equations (8.11) read

$$\left.\begin{aligned} u_3^*(\mu a_\nu a_\nu - \rho b) &= 0, \\ \rho b\sigma^* - u_\nu^* a_\nu &= 0, \\ a_\lambda[U_{\sigma S} S^* + (U_{\sigma\sigma} + \tfrac{1}{3}\mu\rho b)\sigma^*] + u_\lambda^*(\mu a_\nu a_\nu - \rho b) &= 0, \\ \rho T b S^* - \kappa a_\nu a_\nu (U_{SS} S^* + U_{S\sigma}\sigma^*) &= 0. \end{aligned}\right\} \tag{9.2}$$

Elimination of u_λ^* , S^*, σ^* from the last three lines gives the determinantal equation

$$\begin{vmatrix} a_1 & a_2 & 0 & \rho b \\ \rho b - \mu a_r a_r & 0 & a_1 U_{\sigma s} & a_1(U_{\sigma\sigma} + \tfrac{1}{3}\mu\rho b) \\ 0 & \rho b - \mu a_r a_r & a_2 U_{\sigma s} & a_2(U_{\sigma\sigma} + \tfrac{1}{3}\mu\rho b) \\ 0 & 0 & \rho T b - \kappa a_r a_r U_{SS} & -\kappa a_r a_r U_{S\sigma} \end{vmatrix} = 0. \tag{9.3}$$

Introducing the notation (for any axes)

$$A = a_k a_k , \qquad C = \rho b - \mu a_k a_k , \tag{9.4}$$

so that

$$\rho b = C + \mu A, \tag{9.5}$$

and further writing

$$\Pi = \kappa(U_{SS}U_{\sigma\sigma} - U_{\sigma S}^2) - \mu T U_{\sigma\sigma} , \qquad \Sigma = \mu T - \kappa U_{SS} , \tag{9.6}$$

the first of (9.2) and the determinantal equation (9.3) read

$$Cu_3^* = 0, \tag{9.7}$$

$$C(K_3C^3 + K_2C^2 + K_1C + K_0) = 0,$$

where

$$\left.\begin{aligned} K_0 &= A^2(\Pi - \tfrac{1}{3}\mu^2\Sigma A), \\ K_1 &= -ATU_{\sigma\sigma} + \tfrac{1}{3}A^2\mu(\mu T - 2\kappa U_{SS}), \\ K_2 &= A(\tfrac{5}{3}\mu T - \kappa U_{SS}), \\ K_3 &= T. \end{aligned}\right\} \tag{9.8}$$

We note that the complex vector a_i appears now only in the form A, and we shall therefore classify all elementary solutions of the form (8.1) according to the value of A as follows:

Class I: $A \neq 0$, $A \neq 3\,\Pi/(\mu^2\Sigma)$, (9.9)

Class II: $A = 3\,\Pi/(\mu^2\Sigma)$, (9.10)

Class III: $A = 0$. (9.11)

Disturbances of Class I are to be regarded as the general type. By (9.7) we get four values for C, one of which is $C = 0$; hence we get four values for b, one of which is

$$b = \mu A/\rho. \tag{9.12}$$

This means that if we choose the two real vectors a_i' , a_i'' arbitrarily [except for the slight restriction (9.9)], thus fixing the planes of the amplitude waves and the phase waves, then we obtain four values of b, i.e. four values for the velocities of the amplitude

waves and the phase waves. We see, from the first of (9.2), that the solution (9.12) leaves u_3^* arbitrary; the other three values of b make $u_3^* = 0$ and hence give *plane disturbances*, for which also, by (8.9) and (9.1), we have $h_3 = 0$.

A disturbance of Class II makes $K_0 = 0$, and thus, of the four values of C, two are zero. The two non-zero values of C give plane disturbances. Note that in general

$$a_k a_k = a_k' a_k' - a_k'' a_k'' + 2i a_k' a_k'', \tag{9.13}$$

and therefore the reality of A in (9.10) implies the orthogonality of the amplitude waves and the phase waves.

For a disturbance of Class III we have $K_0 = K_1 = K_2 = 0$; therefore $C = 0$ and hence $b = 0$. This is an interesting case, since the vanishing of b tells us that the disturbance is *steady* (independent of time). By (9.13) the vectors a_i' , a_i'' are orthogonal and equal in magnitude, and if we take the axes of x_1 and x_2 along these vectors, the disturbance is of the form

$$\sigma^{(1)} = \sigma^* \exp a_1'(x_1 + ix_2), \tag{9.14}$$

with similar expressions for u_i and $S^{(1)}$. Substituting these expressions in (7.8) or putting $a_1 = 1, a_2 = i, b = 0$ in (9.2), we find that u_i^* , S^*, σ^*, are to satisfy

$$u_1^* + iu_2^* = 0, \qquad U_{\sigma S} S^* + U_{\sigma\sigma}\sigma^* = 0. \tag{9.15}$$

This completes our analysis of elementary disturbances of the form (8.1), which we have here studied for their own sake without reference to boundary conditions. There is of course no implication that such solutions need hold throughout all space and for all time; we may, if we like, consider such solutions (or a superposition of them) in a limited portion of space-time, and investigate what boundary conditions are appropriate to these solutions.

ON AXI-SYMMETRICAL VIBRATIONS OF SHALLOW SPHERICAL SHELLS*

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1. Introduction. The present note may be considered as a sequel to an earlier paper on the same subject [4]. In this earlier paper the solution of the differential equations for axi-symmetrical vibrations of shallow spherical shells was given in terms of certain Bessel functions. The problem of the frequency determination for a shell segment with clamped edge was considered as an example of application of this solution. It led to the vanishing of a third-order determinant each element of which involved Bessel functions and the solution of a certain cubic (Eqs. 32 and 35 of Ref. 4). Similar results, by a somewhat different method, had earlier been obtained by Federhofer [2]. The Bessel-function frequency determinant being difficult to evaluate, no numerical results have yet been obtained by its use. Instead, an approximate solution for the lowest frequency was obtained by means of the procedure of Rayleigh and Ritz (Equation 40 of Ref. 4 and a similar result in Ref. 2).

The Bessel-function solution of Ref. 4 was obtained on the basis of assumptions which had previously been made for the problem of static deformations of shallow spherical shells [3]. It was not observed at that time that an additional approximation would be appropriate for the problem of transverse vibrations. This additional approximation is based on the fact that for transverse vibrations of shallow shells the magnitude of the longitudinal inertia terms is negligibly small compared with the magnitude of the transverse inertia terms [5]. Upon omission of longitudinal inertia terms it becomes possible to reduce the differential equations of dynamics to the same form as the equations of statics except that the transverse load function must include the d'Alembert term $-ph w_{tt}$.

The present paper contains applications of this result to three specific problems of axi-symmetrical vibrations of spherical shells.

(1) *Determination of the lowest frequency of free vibrations for a shell segment with clamped edge.* The numerical results are compared with the corresponding results from the earlier Rayleigh-Ritz formula.

(2) *Determination of the frequencies of free vibrations of a shell segment with free edge.* The relation between the present frequency equation and the corresponding frequency equation for a flat plate is of such nature that known numerical results for the flat plate can be translated with little difficulty so as to furnish the corresponding results for the shallow spherical shell.

(3) *Forced vibrations due to point load at apex of the shell.* We determine first the point-force singularity which occurs regardless of the form of the boundary conditions at the edge of the shell. We then consider further the case for which the boundary is sufficiently far removed to assume it at infinity and use the condition that sufficiently far from the point of load application the solution must represent outward travelling waves.

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Our results generalize corresponding results of H. Cremer and L. Cremer [1] for the unlimited flat plate.

2. Differential equations for transverse vibrations of shallow spherical shells. Let the equation of the shell surface be given by

$$z = \frac{a^2}{2R} - \frac{x^2 + y^2}{2R} = \frac{a^2 - r^2}{2R}. \tag{1}$$

In the absence of longitudinal loads we have as differential equations for the displacement w in the direction of z and for an Airy stress function F,

$$D\nabla^2\nabla^2 w + \frac{1}{R}\nabla^2 F = -\rho h \frac{\partial^2 w}{\partial t^2} + p(r, \theta, t) \tag{2}$$

$$\nabla^2\nabla^2 F - \frac{hE}{R}\nabla^2 w = 0, \tag{3}$$

where ρ = density of shell material, h = wall thickness (assumed constant), E = modulus of elasticity, $D = Eh^3/12(1 - \nu^2)$ and $\nabla^2 = \partial^2/\partial r^2 + r^{-1}\partial/\partial r + r^{-2}\partial^2/\partial\theta^2$ (Fig. 1).

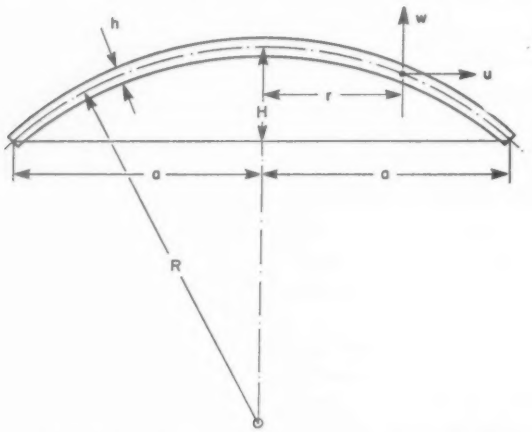

FIG. 1. Spherical shell segment, showing notations for geometrical dimensions.

We shall further need the relations

$$\epsilon_\theta = \frac{N_\theta - \nu N_r}{Eh}, \qquad N_\theta = \frac{\partial^2 F}{\partial r^2}, \qquad N_r = \frac{1}{r}\frac{\partial F}{\partial r} + \frac{1}{r^2}\frac{\partial^2 F}{\partial\theta^2}, \tag{4}$$

$$M_r = -D\left(\frac{\partial^2 w}{\partial r^2} + \frac{\nu}{r}\frac{\partial w}{\partial r} + \frac{\nu}{r^2}\frac{\partial^2 w}{\partial\theta^2}\right), \tag{5}$$

$$Q_r = -D\frac{\partial\nabla^2 w}{\partial r}, \qquad V_r = Q_r + N_r\frac{\partial z}{\partial r} + N_{r\theta}\frac{1}{r}\frac{\partial z}{\partial\theta}, \tag{6}$$

which are the same as in problems of statics.

3. Axi-symmetrical solutions. We set

$$w = e^{i\omega t}W(r), \qquad F = e^{i\omega t}f(r), \qquad p = e^{i\omega t}q(r). \tag{7}$$

Restricting attention to the case $q = 0$ it is found that solutions of the homogeneous equations (2) and (3) are of the form

$$W = C_1J_0(\lambda r) + C_2Y_0(\lambda r) + C_3I_0(\lambda r) + C_4K_0(\lambda r) + C_5 + C_6 \log r, \tag{8}$$

$$\begin{aligned} f = {} & B_1J_0(\lambda r) + B_2Y_0(\lambda r) + B_3I_0(\lambda r) + B_4K_0(\lambda r) \\ & + B_5 r^2 + B_6 r^2 \log r + C_7 + C_8 \log r. \end{aligned} \tag{9}$$

The constants C_n in (8) and (9) are arbitrary, the usual notation for Bessel functions and modified Bessel functions is employed, and furthermore

$$\lambda^4 = \frac{\rho h}{D}\omega^2 - \frac{hE}{R^2D}. \tag{10}$$

Taking account of the fact that

$$\nabla^2(J_0 , Y_0) = -\lambda^2(J_0 , Y_0), \qquad \nabla^2(I_0 , K_0) = \lambda^2(I_0 , K_0), \tag{11}$$

there results for the constants B_n

$$\left.\begin{aligned} B_{1,2} &= -\frac{hE}{R\lambda^2}C_{1,2} \\ B_{3,4} &= \frac{hE}{R\lambda^2}C_{3,4} \\ B_5 &= \frac{1}{4}\rho h\omega^2 R(C_5 - C_6) \\ B_6 &= \frac{1}{4}\rho h\omega^2 RC_6 \end{aligned}\right\}. \tag{12}$$

Of the eight constants C_n only six are physically significant. The constant C_7 has no effect on stresses and displacements. The condition of vanishing circumferential displacements as in problems of statics [3], is of the form $(hE/R)W = \nabla^2 f + \text{const}$. From this $(hE/R)C_6 = 4B_6$. With B_6 from (12) this implies $\lambda^4 C_6 = 0$. Since $\lambda \neq 0$ we have

$$B_6 = C_6 = 0 \tag{12'}$$

and therewith altogether

$$W = C_1J_0(\lambda r) + C_2Y_0(\lambda r) + C_3I_0(\lambda r) + C_4K_0(\lambda r) + C_5 , \tag{13}$$

$$f = -\frac{hE}{R\lambda^2}[C_1J_0(\lambda r) + C_2Y_0(\lambda r) - C_3I_0(\lambda r) - C_4K_0(\lambda r)] + \frac{1}{4}\rho h\omega^2 RC_5 r^2 + C_8 \log r. \tag{14}$$

The solution (13) and (14) will be applied to the three problems indicated earlier.

4. Frequency equation for shell segment with clamped edge. Let $r = a$ be the edge of the shell segment. Since W and f must be regular for $r = 0$ we have

$$C_2 = C_4 = C_8 = 0. \tag{15}$$

The conditions of vanishing edge displacement W and edge slope W' are

$$C_1 J_0(\mu) + C_3 I_0(\mu) + C_5 = 0, \tag{16}$$

$$C_1 J_0'(\mu) + C_3 I_0'(\mu) = 0, \tag{17}$$

where

$$\mu = \lambda a. \tag{18}$$

We assume that the third edge condition stipulates vanishing horizontal displacement or, equivalently, vanishing circumferential strain ϵ_ϑ . With the help of (4) this relation becomes $(F'' - \nu r^{-1}F')_a \equiv [\nabla^2 F - (1 + \nu) r^{-1} F']_a = 0$. Introduction of (14) gives as third relation for the constants C_1 , C_3 , C_5 ,

$$\frac{hE}{R}\left\{C_1 J_0(\mu) + C_3 I_0(\mu) - \frac{1+\nu}{\mu}\left[-C_1 J_0'(\mu) + C_3 I_0'(\mu)\right]\right\} + \frac{1}{2}\rho h\omega^2 R(1-\nu)C_5 = 0. \tag{19}$$

The frequency equation of the problem is the condition of vanishing of the determinant of the system (16), (17) and (19). Setting $J_0' = -J_1$, $I_0' = I_1$, this frequency equation may be brought into the following form

$$J_0(\mu) I_1(\mu) + J_1(\mu) I_0(\mu) + \frac{4\kappa^4 J_1(\mu) I_1(\mu)}{\mu(\mu^4 - \kappa^4)} = 0. \tag{20}$$

The parameter κ^4 is

$$\kappa^4 = \frac{1+\nu}{1-\nu}\frac{hEa^4}{R^2 D} = 48(1+\nu)^2 \frac{H^2}{h^2} \tag{21}$$

and ω is given in terms of μ and κ, as follows

$$\rho\omega^2 \frac{ha^4}{D} = \frac{1-\nu}{1+\nu}\kappa^4 + \mu^4. \tag{22}$$

The quantity H in (21) is the rise of the shell. It is related to the radii R and a through the formula,

$$H = \frac{a^2}{2R}. \tag{23}$$

The frequency equation (20) furnishes μ as a function of κ. Approximate values of μ for the lowest frequency may be found in Table 1.* Having μ as function of κ we obtain further from (22)

$$\omega = \left(\frac{E}{\rho}\right)^{1/2} \frac{h}{a^2}\left[\frac{\mu^4 + \kappa^4(1-\nu)/(1+\nu)}{12(1-\nu^2)}\right]^{1/2}. \tag{24}$$

Let ω_0 be the value of ω corresponding to $\kappa = 0$ and $\nu = 0$, that is the value of the frequency ω for a flat plate and for vanishing Poisson's ratio. We have

$$\omega_0 = \left(\frac{E}{\rho}\right)^{1/2} \frac{h}{a^2}\left(\frac{\mu_0^4}{12}\right)^{1/2} = 2.948\left(\frac{E}{\rho}\right)^{1/2}\frac{h}{a^2}, \tag{25}$$

*For these and all other computations in this paper the author is indebted to Millard W. Johnson.

TABLE 1. Numerically smallest solution of Eq. (20)

κ^4	μ
0	3.196
10	3.235
20	3.273
50	3.380
100	3.537
215.56	3.832
300	4.000
400	4.180
500	4.328
600	4.460
700	4.575
800	4.680
900	4.770
1000	4.855
1200	4.990
1400	5.112
1600	5.212
1800	5.285
2000	5.360
2500	5.482
3000	5.570
4000	5.665
5000	5.723
10000	5.828
15900	5.865
21150	5.875
∞	5.910

where $\mu_0 = 3.195$ and $\mu_0^4 = 104.27$. It is convenient to express the actual values of ω in units of ω_0, as follows,

$$\frac{\omega}{\omega_0} = \left[\frac{1}{1-\nu^2}\left(\frac{\mu}{\mu_0}\right)^4 + \frac{48}{\mu_0^4}\left(\frac{H}{h}\right)^2\right]^{1/2}. \tag{26}$$

Numerical values of ω/ω_0 as a function of ν and H/h may be found in Table II and in Fig. 2.

When H/h is large enough so that the second term under the square root in (26) dominates the first term, practically when H/h is larger than about 25, then Eq. (26) may be replaced by the approximation

$$\omega = 2\left(\frac{E}{\rho}\right)^{1/2}\frac{H}{a^2}. \tag{26*}$$

It is interesting to note that the thickness h which occurs in the flat-plate frequency formula (25) has been replaced by the shell rise H and the frequency has become independent of wall thickness. It should be emphasized, however, that the simple formula (26*) depends on two limitations. On the one hand we must have that $H/h \gtrsim 25$ and on the other hand we must have $H/a \lesssim \frac{1}{4}$ in order that the theory of shallow shells is

TABLE 2. Frequency of clamped-edge shell in units of corresponding flat-plate frequency for $\nu = 0$, as function of ν and H/h

$\frac{H}{h}$	ω/ω_0		
	$\nu = 0$	$\nu = .3$	$\nu = .5$
0	1.000	1.0483	1.1547
.5	1.08	1.149	1.28
1.0	1.31	1.40	1.59
1.5	1.61	1.75	2.00
2.0	1.94	2.16	2.42
2.5	2.31	2.57	2.87
3.0	2.67	2.99	3.32
3.5	3.06	3.40	3.73
4.0	3.43	3.78	4.12
4.5	3.81	4.16	4.49
5	4.18	4.51	4.89
6	4.90	5.17	5.45
7	5.59	5.80	6.01
8	6.22	6.41	6.60
9	6.86	7.01	7.20
10	7.50	7.62	7.82
11	8.13	8.25	8.44
12	8.76	8.88	9.05
14	10.05	10.15	10.28
16	11.35	11.42	11.54
18	12.67	12.73	12.82
20	14.00	14.1	14.2

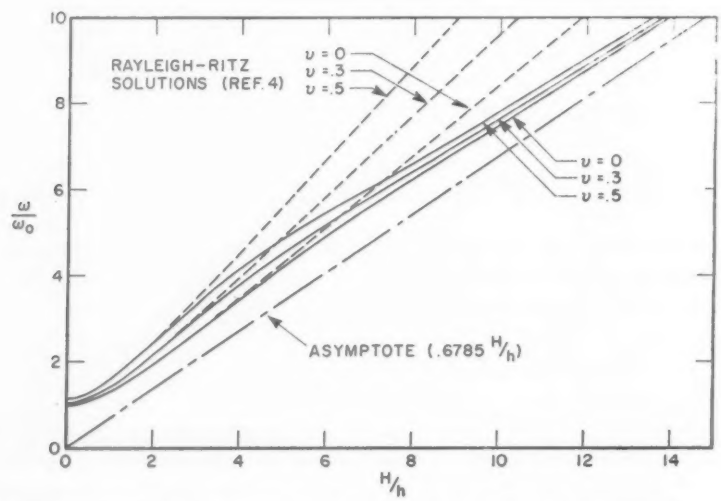

FIG. 2. Lowest frequency for shell segment with clamped edge, in units of corresponding frequency for flat plate with zero Poisson's ratio.

applicable. We may note that it is this latter restriction which insures that the transverse vibrations of the shell take place at frequencies which are low compared with the frequencies of longitudinal vibrations of flat plates, these latter being of order $(E/\rho)^{1/2}(1/a)$.

Comparison with Rayleigh-Ritz formula. Equation (40) of Ref. 4 may be written in the following form, which is equivalent to (26)

$$\left[\frac{\omega}{\omega_0}\right]_{RR} = \left[\frac{.1}{1-\nu^2} + \frac{.7 + .5\nu - .2\nu^2}{1-\nu^2}\left(\frac{H}{h}\right)^2\right]^{1/2}. \tag{27}$$

Figure 2 contains values of $(\omega/\omega_0)_{RR}$ as dotted lines. It is seen that the Rayleigh-Ritz solution agrees remarkably well with the differential equation solution for sufficiently small values of H/h, practically up to values of H/h of about three. As H/h increases the error becomes larger, the percentage error approaching a finite limiting value of from 60 per cent to 25 per cent as ν decreases from 0.5 to 0.

5. Frequency equation for shell segment with free edge. The boundary conditions for a free edge are

$$M_r = 0, \qquad V_r = 0, \qquad N_r = 0, \tag{28}$$

where M_r, V_r and N_r are given in (4) to (6). To this are again added regularity conditions for $r = 0$ which again mean that Eq. (15) must hold. We observe further that (28) contains two relations involving W', W'' and W''' but neither W itself nor f and its derivatives. This means that the frequency equation for the shell with free edge does not involve the constant C_5 in Eq. (13) for W. In other words the admissible values of μ are the same as those for a flat plate with free edge.

The frequency equation for axially symmetric vibrations of a flat plate with free edge, according to Kirchhoff, is

$$\frac{I_0(\mu)}{I_1(\mu)} + \frac{J_0(\mu)}{J_1(\mu)} = \frac{2(1-\nu)}{\mu}. \tag{29}$$

Examples of the numerically smallest values of μ are $\nu = 0$, $\mu = 2.87$; $\nu = .3$, $\mu = 3.00$; $\nu = .5$, $\mu = 3.07$. Having μ we obtain ω by means of (10), (18) and (23) in the form

$$\omega = \left(\frac{E}{\rho}\right)^{1/2}\left[\frac{\mu^4}{12(1-\nu^2)}\frac{h^2}{a^4} + 4\frac{H^2}{a^4}\right]^{1/2}. \tag{30}$$

Let μ_0 and ω_0 be appropriate values for the flat plate and for $\nu = 0$. We may then write, in analogy to the result (26) for the shell with clamped edge,

$$\frac{\omega}{\omega_0} = \left[\frac{1}{1-\nu^2}\left(\frac{\mu}{\mu_0}\right)^4 + \frac{48}{\mu_0^4}\left(\frac{H}{h}\right)^2\right]^{1/2}. \tag{31}$$

Equation (31) is the same as Eq. (26) except that for the shell with free edge the quantity μ is independent of the values of H/h while for the shell with clamped edge μ was found to be a function of H/h. Values of ω/ω_0 for the numerically smallest solution of (29) as function of H/h and ν may be found in Table 3 and Fig. 3.

We note that when H/h is greater than about 10, Eq. (30) may be simplified to $\omega = 2(E/\rho)^{1/2}(H/a^2)$ which is the same limiting expression which was previously obtained for the shell segment with clamped edge.

We further note that very probably for this problem of a shell with free edge the

TABLE 3. Frequency of free-edge shell in units of corresponding flat-plate frequency for $\nu = 0$, as function of ν and H/h

$\frac{H}{h}$	ω/ω_0		
	$\nu = 0$	$\nu = .3$	$\nu = .5$
0	1	1.148	1.315
.5	1.084	1.224	1.380
1.0	1.305	1.423	1.562
1.5	1.610	1.705	1.821
2.0	1.96	2.035	2.135
2.5	2.33	2.40	2.48
3.0	2.71	2.77	2.85
3.5	3.11	3.16	3.22
4.0	3.51	3.55	3.62
4.5	3.91	3.96	4.01
5	4.32	4.36	4.41
6	5.15	5.17	5.21
7	5.98	6.00	6.03
8	6.81	6.82	6.86
9	7.64	7.66	7.70
10	8.47	8.49	8.53
11	9.31	9.32	9.35
12	10.15	10.15	10.18
14	11.82	11.82	11.82
16	13.50	13.50	13.50
18	15.20	15.20	15.20
20	16.86	16.86	16.86

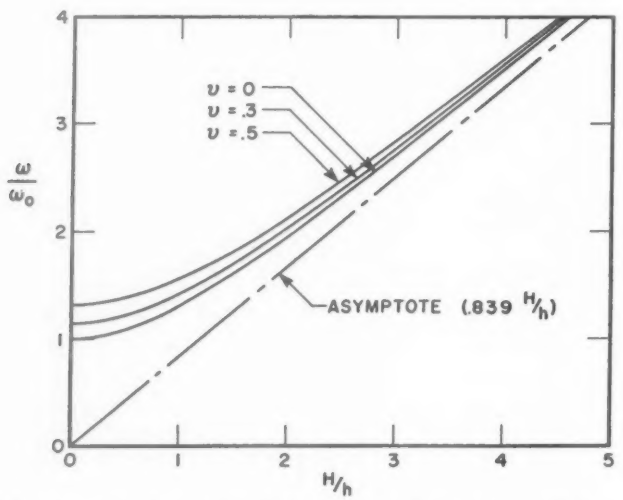

FIG. 3. Lowest frequency of axi-symmetrical vibrations for shell segment with free edge, in units of corresponding frequency for flat plate with zero Poisson's ratio.

lowest frequency of axi-symmetrical free vibrations is higher than the frequencies of certain non-symmetrical vibrations. This fact is concluded from corresponding known results for the special case of the flat plate.

6. Oscillating point load at apex of shell. The following conditions must be satisfied for an oscillating point load $P \exp(i\omega t)$ at the apex of the shell

$$\lim_{r\to 0} (2\pi r V_r) = Pe^{i\omega t}, \tag{32}$$

$$w(0, t), \qquad N_r(0, t), \qquad N_\theta(0, t) \qquad \text{finite.} \tag{33}$$

In view of (4), (5), (6), (7) these conditions assume the following form for the solution functions W and f as given by (13) and (14)

$$\lim_{r\to 0} \left(r \frac{d\nabla^2 w}{dr} \right) = -\frac{P}{2\pi D}, \tag{32'}$$

$$r = 0; \qquad W, \nabla^2 f, \qquad \frac{1}{r}\frac{df}{dr} \qquad \text{finite.} \tag{33'}$$

We begin by omitting terms from the solutions (13) and (14) which automatically satisfy the conditions (32') and (33') and retain the singular portion

$$W_s = C_2 Y_0(\lambda r) + C_4 K_0(\lambda r), \tag{34}$$

$$f_s = -\frac{hE}{R\lambda^2} [C_2 Y_0(\lambda r) - C_4 K_0(\lambda r)] + C_8 \log r. \tag{35}$$

Equations (34) and (35) are valid, with real values of the constants C_2 and C_4, as long as λ is real. According to (10) this means as long as

$$\frac{1}{R}\left(\frac{E}{\rho}\right)^{1/2} < \omega. \tag{36a}$$

When

$$\omega < \frac{1}{R}\left(\frac{E}{\rho}\right)^{1/2} = \frac{2H}{a^2}\left(\frac{E}{\rho}\right)^{1/2} \tag{36b}$$

the representations (34) and (35) are no longer convenient. It will, however, be shown that the results obtained in the range of frequencies (36a) are readily transferred to the range (36b).

We note that the frequency which divides the ranges (36a) and (36b) is exactly that lowest frequency of *free* vibrations which occurs in the range of sufficiently large values of H/h [See Eq. (26*)].

In order to determine the constants C_2, C_4 and C_8 we observe the following relations in which use has been made of (11),

$$\nabla^2 W_s = \lambda^2[-C_2 Y_0 + C_4 K_0], \tag{37a}$$

$$\nabla^2 f_s = \frac{hE}{R}[C_2 Y_0 + C_4 K_0], \tag{37b}$$

$$\frac{1}{r}\frac{df_s}{dr} = -\frac{hE}{R}\frac{1}{\lambda r}[C_2 Y_0' - C_4 K_0'] + \frac{C_8}{r^2}, \tag{37c}$$

$$r\frac{d\nabla^2 W_s}{dr} = \lambda^2(\lambda r)[-C_2 Y_0' + C_4 K_0'], \tag{37d}$$

$$\left.\begin{aligned} Y_0(\lambda r) &\approx \frac{2}{\pi}\left[\log\frac{\lambda r}{2} + .577\cdots\right], \qquad Y_0'(\lambda r) \approx \frac{2}{\pi}\frac{1}{\lambda r} \\ K_0(\lambda r) &\approx -\left[\log\frac{\lambda r}{2} + .577\cdots\right], \qquad K_0'(\lambda r) \approx -\frac{1}{\lambda r} \end{aligned}\right\}, \tag{38}$$

Eqs. (38) being valid when $\lambda r \ll 1$.

Introduction of (34), (35), (37) and (38) into the finiteness conditions (33′) shows that these conditions are satisfied provided

$$\frac{2}{\pi} C_2 - C_4 = 0, \tag{39a}$$

$$\frac{2}{\pi} C_2 + C_4 = \frac{\lambda^2 R}{Eh} C_8 . \tag{39b}$$

The load condition (32′) leads to the further relation

$$\frac{2}{\pi} C_2 + C_4 = \frac{P}{2\pi D\lambda^2}. \tag{39c}$$

Introduction of C_2, C_4 and C_8 from (39) into (34) and (35) leads to the following expressions for the singular solutions W_s and f_s,

$$W_s = \frac{P}{4\pi D\lambda^2}\left[\frac{\pi}{2} Y_0(\lambda r) + K_0(\lambda r)\right], \tag{40}$$

$$f_s = -\frac{Eh}{R\lambda^2}\frac{P}{4\pi D\lambda^2}\left[\frac{\pi}{2} Y_0(\lambda r) - K_0(\lambda r) - 2\log r\right]. \tag{41}$$

It will now be shown that (40) and (41) remain valid, but are conveniently written in different form, in the range of small ω given by (36b). For values of ω in the range (36b) we write

$$\lambda = i^{1/2}\left[\frac{hE}{R^2 D} - \frac{\rho h\omega^2}{D}\right]^{1/4} \equiv i^{1/2}\gamma \tag{42}$$

and we introduce Kelvin functions through the following known relations

$$Y_0(i^{1/2}\gamma r) = \frac{2}{\pi}[-\ker\gamma r + i\,\mathrm{kei}\,\gamma r], \tag{43a}$$

$$K_0(i^{1/2}\gamma r) = \ker\gamma r + i\,\mathrm{kei}\,\gamma r. \tag{43b}$$

If (42) and (43) are introduced into (40) and (41) there follow as expressions for W_s and f_s which are valid when $\omega < R^{-1}(E/\rho)^{1/2}$,

$$W_s = \frac{P}{2\pi D\gamma^2}\,\mathrm{kei}\,\gamma r, \tag{44}$$

$$f_s = -\frac{Eh}{R\gamma^2}\frac{P}{2\pi D\gamma^2}[\ker\gamma r + \log r]. \tag{45}$$

Equations (44) and (45) contain as special case, when $\omega = 0$, the previously given formulas [3, Eq. (48)] for the corresponding problem of static deflection. Writing

$\gamma = (hE/R^2D)^{1/4} = [12(1 - \nu^2)]^{-1/4}(Rh)^{-1/2}$ Eqs. (44) and (45) become for $\omega = 0$

$$W_s = \frac{[12(1 - \nu^2)]^{1/2}}{2\pi}\frac{PR}{Eh}\,\mathrm{kei}\,\frac{r}{[12(1 - \nu^2)]^{1/4}(Rh)^{1/2}}, \tag{44'}$$

$$f_s = -\frac{PR}{2\pi}\left[\ker\frac{r}{[12(1 - \nu^2)]^{1/4}(Rh)^{1/2}} + \log r\right]. \tag{45'}$$

We finally note the following important fact. As long as the solution (44) applies, that is as long as $\omega < (2H/a^2)(E/\rho)^{1/2}$, we have that the deflection amplitude W decreases exponentially at large distances r from the point of load application. When ω is larger than $(2H/a^2)(E/\rho)^{1/2}$ so that the solution (40) applies then the deflection amplitude W decreases as $r^{-1/2}$, that is, much more slowly than exponentially. In this latter case an important distinction can be made between standing wave and travelling wave solutions.

7. Travelling wave solution for effectively unlimited shell segment. We inquire for a solution in which the deflection $w = \exp(i\omega t)W(r)$ behaves for sufficiently large values of r as $r^{-1/2}\exp[i(\omega t - \lambda r)]$. For a solution with this type of behavior we have that the wave produced by the pulsating point load travels *outward*, with energy being dissipated through radiation.

In view of the asymptotic behavior of the Hankel function of the second kind

$$H_0^{(2)}(\lambda r) = J_0(\lambda r) - iY_0(\lambda r) \sim \left(\frac{2}{\pi\lambda r}\right)^{1/2}\exp[-i(\lambda r - \pi/4)], \tag{46}$$

an appropriate result is obtained by adding in (40) and (41) a suitable multiple of the non-singular solution $J_0(\lambda r)$ as follows,

$$W_{ST} = \frac{P}{4\pi D\lambda^2}\left\{\frac{\pi}{2}[Y_0(\lambda r) + iJ_0(\lambda r)] + K_0(\lambda r)\right\}, \tag{47}$$

$$f_{ST} = -\frac{Eh}{R\lambda^2}\frac{P}{4\pi D\lambda^2}\left\{\frac{\pi}{2}[Y_0(\lambda r) + iJ_0(\lambda r)] - K_0(\lambda r) - 2\log r\right\}. \tag{48}$$

We note that the solution (47) and (48) contains as a special case the corresponding result for a flat plate, as given by H. Cremer and L. Cremer [1]. The case of the flat plate follows if we set $R = \infty$ in the expression for λ, making $\lambda = (\rho h\omega^2/D)^{1/4}$. We note further that while for the spherical shell the solution (47) and (48) is valid, subject to the restriction that $\omega > (E/\rho)^{1/2}R$, this relation ceases to be a restriction for the case of the flat plate.

In order that the travelling wave solution (47) have a meaning it is necessary that the boundary $r = a$ be sufficiently far removed from the point of load application. Let us assume that sufficiently far means

$$a\lambda > 5, \qquad \text{say.} \tag{49}$$

Equation (49) may be written as a restriction on frequencies as follows

$$\omega > \frac{2H}{a^2}\left(\frac{E}{\rho}\right)^{1/2}\left[1 + \frac{625}{48(1 - \nu^2)}\frac{h^2}{H^2}\right]^{1/2}. \tag{49'}$$

8. Two formulas of acoustical significance. We consider the ratio of velocity at the point of load application to the force causing this velocity, $(\partial w/\partial t)_{r=0}/P$ **exp** $(i\omega t)$.

According to Eq. (47)

$$\frac{(\partial w/\partial t)_{r=0}}{Pe^{i\omega t}} = \frac{i\omega}{4\pi D\lambda^2} \lim_{r\to 0} \left\{ \frac{\pi}{2} [Y_0(\lambda r) + iJ_0(\lambda r)] + K_0(\lambda r) \right\}.$$

Since $J_0(0) = 1$ and, according to (38), $\lim_{r\to 0} [\frac{1}{2}\pi Y_0(\lambda r) + K_0(\lambda r)] = 0$ there follows,

$$\frac{(\partial w/\partial t)_{r=0}}{Pe^{i\omega t}} = \frac{-\omega}{8D\lambda^2} = \frac{-\omega}{8\left[\rho h D\left(\omega^2 - 4\frac{E}{\rho}\frac{H^2}{a^4}\right)\right]^{1/2}}, \qquad \frac{2H}{a^2}\left(\frac{E}{\rho}\right)^{1/2} < \omega. \tag{50a}$$

For $H = 0$, Eq. (50a) reduces to a result in Ref. 1. We note that a ratio which is independent of frequency for the flat plate is a function of frequency for the shallow spherical shell. In the range of applicability of (50a), which is given by (49′), this ratio decreases with increasing frequency towards a limiting value which is the constant flat plate value.

When $\omega < (2H/a^2)(E/\rho)^{1/2}$ so that Eq. (44) applies we have instead of (50a)

$$\frac{(\partial w/\partial t)_{r=0}}{Pe^{i\omega t}} = \frac{i\omega \mathrm{kei}(0)}{2\pi D\gamma^2}.$$

In view of the fact that $\mathrm{kei}(0) = -\frac{1}{4}\pi$ and with γ defined in (42) this may be written in the following form

$$\frac{(\partial w/\partial t)_{r=0}}{Pe^{i\omega t}} = \frac{-i\omega}{8\left[\rho h D\left(4\frac{E}{\rho}\frac{H^2}{a^4} - \omega^2\right)\right]^{1/2}}, \qquad \frac{2H}{a^2}\left(\frac{E}{\rho}\right)^{1/2}. \tag{50b}$$

We finally observe, on the basis of (50a) and (50b) that the following result holds for the work of the force $P \exp(i\omega t)$ per cycle

$$\frac{\text{Work}}{\text{Cycle}} = \begin{cases} 0, & \omega < \dfrac{2H}{a^2}\left(\dfrac{E}{\rho}\right)^{1/2}, \\ \dfrac{P^2}{4\left[\rho h D\left(\omega^2 - 4\dfrac{E}{\rho}\dfrac{H^2}{a^4}\right)\right]^{1/2}}, & \dfrac{2H}{a^2}\left(\dfrac{E}{\rho}\right)^{1/2} < \omega. \end{cases} \tag{51}$$

It is recalled that Eq. (51) is derived without consideration of damping sources other than radiation damping. In addition to this, while formally the second part of Eq. (51) holds for all ω greater than $(2H/a^2)(E/\rho)^{1/2}$, for the solution to have physical meaning a stronger restriction such as (49′) actually applies.

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SCATTERING OF ELECTROMAGNETIC WAVES FROM A RANDOM SURFACE*

BY

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Abstract. Suppose $Z(x, y)$ is a real random process, continuous in the mean over a finite region D, and with mean value zero and covariance function $r(x, y; x', y')$. As such $Z(x, y)$ defines a random surface (provided certain differentiability conditions are satisfied). Then, by Karhunen's theorem on the representation of a random function, $Z(x, y)$ has an expansion in terms of an orthogonal process and the eigenfunctions and eigenvalues of the covariance function. Introducing this expression into the far-zone form of the Stratton-Chu solution of the electromagnetic field equations then leads to an approximate expression for the radiation scattered from the random surface, from which mean and covariance of the scattered field can be determined.

1. Introduction. The problem of determining the radiation scattered from a surface which varies randomly about some mean surface, for example a plane, leads to some interesting mathematical problems. The theoretical treatment is complicated by the fact that the boundary conditions are functionals of the random function describing the boundary, and a general theory of such boundary value problems appears to be lacking. Even in the relatively simple case of a perfectly conducting surface the boundary conditions themselves are subject to an integral equation[1] of apparently new type. The formidable difficulties in relating the field to the random boundary are, however, eased somewhat by assuming the medium below the random surface to be perfectly conducting, and this will be supposed in what follows. One can then avail himself of an approximate form of the Stratton-Chu solution, called the "current distribution" method, based on the "Kirchoff approximation", i.e., the assumption that the incident electromagnetic wave is reflected at every point as though an infinite plane wave were incident upon the infinite tangent plane. This requires that the curvature of the surface everywhere be small, and further that the incident plane wave not be at grazing incidence.

The spirit of the present paper is matched most closely by some work of Isakovich[2], who also employed the current distribution method to study scattering from a random surface. Isakovich's treatment, however, ignored the edge effects, and the probabilistic development was purely formal. Previous investigations[3,4,5] have in general not gone very deeply into the probabilistic aspects of the problem. A recent paper on a related subject, the solution of Laplace's equation in a half-plane with boundary conditions consisting of random functions, by Kampé de Feriet[6] is, however, a notable exception.

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2. Definition of the random surface. Suppose that $Z(x, y)$ is a separable random process, real and continuous in the mean over a finite region D, with mean zero and covariance function $r(x, y; x', y')$. Since $Z(x, y)$ is to represent a surface, we require it to be three times mean square differentiable. This is assured if the covariance function has partial derivatives up to and including the fourth order[7,8]. Further, since the curvature of the surface must be small if the current distribution method is to apply, it will be supposed that this is true for almost all realizations[9].

The surface $Z(x, y)$ has been supposed to be a separable random function. Now this is the case if and only if[10] there exists the bilinear representation:

$$r(x, y; x', y') = \sum_{m,n=1}^{\infty} (1/\lambda_{mn})\varphi_{mn}(x, y)\varphi_{mn}(x', y'), \tag{1}$$

where the $\varphi_{mn}(x, y)$ and λ_{mn} are the eigenfunctions and eigenvalues of the integral equation:

$$\varphi(x, y) = \lambda \iint_D r(x, y; x', y')\varphi(x', y')\, dx'\, dy'. \tag{2}$$

Then according to Karhunen's representation theorem[10] we have for every $(x, y) \in D$ the expansion

$$Z(x, y) = \text{l.i.m.} \sum_{m,n} \lambda_{mn}^{-1/2}\varphi_{mn}(x, y)z_{mn}\ , \tag{3}$$

in terms of the orthogonal process $\{z_{mn}\}$ with[11]

$$\begin{aligned} \mathcal{E}z_{mn} &= 0, \\ \mathcal{E}\{z_{mn}z_{pq}\} &= \begin{cases} 0, & \text{if } m \neq n \text{ and/or } p \neq q; \\ 1, & \text{if } m = n \text{ and } p = q. \end{cases} \end{aligned} \tag{4}$$

At this point we shall assemble certain relations which will be used repeatedly in the sequel. Let the characteristic function of the function $f_{mn}(x, y)$ with respect to the random variable z_{mn} be denoted by

$$\chi[f_{mn}(x, y)] = \mathcal{E}\{\exp\,[if_{mn}(x, y)z_{mn}]\}, \tag{5}$$

and introduce the abbreviated notations:

$$B_{mn} = B_{mn}(x, y) = kB\lambda_{mn}^{-1/2}\varphi_{mn}(x, y) \tag{6}$$

and

$$C_{mn} = C_{mn}(x, y; x', y') = k\lambda_{mn}^{-1/2}[B_1\varphi_{mn}(x, y) - B_2\varphi_{mn}(x', y')], \tag{7}$$

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[9]It seems to be a difficult matter to establish satisfactory sufficient conditions equivalent to this hypothesis.

[10]K. Karhunen, *Über lineare Methoden in der Wahrscheinlichkeitsrechnung*, *Ann. Acad. Scient. Fennicae* **(A) 37**, 1-79 (1947).

[11]The mathematical expectation of the random variable ξ will be denoted by $\mathcal{E}\xi$. The letter E will be reserved for the electric field.

where

$$B = B(\alpha); \qquad B_1 = B(\alpha_1); \qquad B_2 = B(\alpha_2),$$

and α, α_1 , α_2 are certain angles introduced in Sec. 3 below.

Then the following expressions hold in the sense of mean covergence:

$$\mathcal{E}\{\exp [ikBZ(x, y)]\} = \prod_{m,n} \chi(B_{mn}) = \chi(kB; x, y). \tag{8}$$

$$\mathcal{E}\{z_{mn} \exp [ikB \sum_{p,q} \lambda_{pq}^{-1/2} \varphi_{pq} z_{pq}]\} = -i \frac{d}{dB_{mn}} [\prod_{p,q} \chi(B_{pq})]. \tag{9}$$

and

$$\mathcal{E}\{\exp (ik[B_1 Z(x, y) - B_2 Z(x', y')])\} = \prod_{m,n} \chi(C_{mn}). \tag{10}$$

$$\mathcal{E}\{z_{mn} \exp [i \sum_{p,q} C_{pq} z_{pq}]\} = -i \frac{d}{dC_{mn}} (\prod_{p,q} \chi(C_{pq})). \tag{11}$$

$$\mathcal{E}\{z_{mn} z_{pq} \exp [i \sum_{s,t} C_{st} z_{st}]\} = \begin{cases} -\dfrac{d^2\chi(C_{mn})}{dC_{mn}^2} \prod' \chi(C_{st}), & m = p \quad \text{and} \quad n = q; \\ -\dfrac{d\chi(C_{mn})}{dC_{mn}} \dfrac{d\chi(C_{pq})}{dC_{pq}} \prod'' \chi(S_{st}), & \\ \qquad\qquad \text{when } p \neq m \text{ and/or } q \neq n. \end{cases} \tag{12}$$

The notation Π' is to indicate that the product is to be taken over $s \neq m$ and/or $t \neq n$; similarly Π'' is to mean a product over $s \neq m$, p and $t \neq n$, q. It is of course understood that the derivatives indicated above are purely formal only. The expressions (8)-(12) are easily obtained from (3) and (4) and the notations (5)-(7).

If $Z(x, y)$ is a Gaussian process with mean zero and variance $\sigma^2(x, y)$, the above expressions reduce to

$$\begin{aligned} \mathcal{E}\{\exp [ikBZ(x, y)]\} &= \exp \{-\tfrac{1}{2}k^2B^2 \sum \lambda_{mn}^{-1} \varphi_{mn}^2(x, y)\} \\ &= \exp \{-\tfrac{1}{2}k^2B^2\sigma^2(x, y)\}. \end{aligned} \tag{13}$$

$$\mathcal{E}\{z_{mn} \exp [ikB \sum_{p,q} \lambda_{pq}^{-1/2} \varphi_{pq}(x, y) z_{pq}]\} = ikB\lambda_{mn}^{-1/2} \varphi_{mn}(x, y) \exp \{-\tfrac{1}{2}k^2B^2\sigma^2(x, y)\} \tag{14}$$

and

$$\mathcal{E}\{\exp (ik[B_1 Z(x, y) - B_2 Z(x', y')])\} = \exp (-\tfrac{1}{2}k^2Q) \tag{15}$$

$$\mathcal{E}\{z_{mn} \exp [i \sum_{p,q} C_{pq} z_{pq}]\} = iC_{mn} \exp (-\tfrac{1}{2} \sum_{p,q} C_{pq}^2) \tag{16}$$

$$\mathcal{E}\{z_{mn} z_{pq} \exp (i \sum C_{st} z_{st})\} = \exp (-\tfrac{1}{2}k^2Q) \begin{cases} M_{mn}(x, y; x', y'), & \\ \quad \text{provided} \quad m = p \quad \text{and} \quad n = q; & \\ M'_{mnpq}(x, y; x', y'), & \\ \quad \text{if} \quad p \neq m \quad \text{and/or} \quad q \neq n, & \end{cases} \tag{17}$$

where

$$Q = Q(x, y; x', y') = B_1^2\sigma^2(x, y) + B_2^2\sigma^2(x', y') - 2B_1B_2r(x, y; x', y'); \tag{18}$$

$$M_{mn}(x, y; x', y') = 1 - (k^2/\lambda_{mn})[B_1^2\varphi_{mn}^2(x, y) + B_2^2\varphi_{mn}^2(x', y') - 2B_1B_2\varphi_{mn}(x, y)\varphi_{mn}(x', y')]; \tag{19}$$

$$M'_{mnpq}(x, y; x', y') = -\frac{k^2}{(\lambda_{mn}\lambda_{pq})^{1/2}}\{B_1^2\varphi_{mn}(x, y)\varphi_{pq}(x, y) + B_2^2\varphi_{mn}(x', y')\varphi_{pq}(x', y') - B_1B_2[\varphi_{mn}(x, y)\varphi_{pq}(x', y') + \varphi_{mn}(x', y')\varphi_{pq}(x, y)]\}. \tag{20}$$

3. The scattered field. The far-zone electromagnetic field at point P of the radiation scattered from the surface $\Sigma = \{Z(x, y) \mid (x, y) \in D\}$ is given by [12].

$$\mathbf{E}_s(P) = -\frac{i\omega\mu}{2\pi R}\exp(-ikR)\int_\Sigma \{\mathbf{n} \times \mathbf{H}_0 - [(\mathbf{n} \times \mathbf{H}_0)\cdot\mathbf{R}]\mathbf{R}\}\exp(ik\mathbf{d}\cdot\mathbf{R})\,dS, \tag{1}$$

where R is the distance from the origin of coordinates on Σ to the field point P; $\mathbf{R}$ is a unit vector in the direction of R; $\mathbf{n}$ is the unit normal vector to the surface Σ, with the same sense as the z-axis; $\mathbf{d}$ is as shown in Fig. 1; and $\mathbf{H}_0$ is the magnetic vector of the

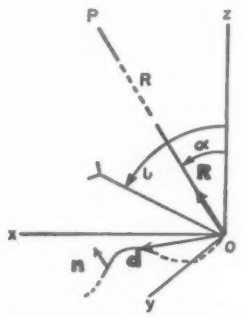

FIG. 1. Coordinate system for far-zone scattered field.

incident electromagnetic field. Expression (1) represents the Stratton-Chu solution[13] of the electromagnetic field equations in the far-zone after the introduction of a line distribution of charge on the boundary. Thus the edge effect of the shadowed area of Σ has been removed.

The fact that (1) represents the far-zone field is usually taken to mean that its applicability is limited by the condition $R > 2\delta^2(D)/\lambda$, where $\delta(D)$ is the diameter of the set D and λ is the wave length of the radiation.

Further progress is dependent upon specification of the magnetic vector $\mathbf{H}_0$ of the incident field. An incident *plane wave* will be assumed, so that for *vertical polarization*,

$$\mathbf{H}_0 = \mathbf{j}(E_0/\eta)\exp\{i[\omega t + k(x\sin\iota + Z\cos\iota)]\}, \qquad Z \in \Sigma, \tag{2}$$

and for *horizontal polarization*,

$$\mathbf{H}_0 = (\mathbf{i}\cos\iota - k\sin\iota)(E_0/\eta)\exp\{i[\omega t + k(x\sin\iota + Z\cos\iota)]\}, \qquad Z \in \Sigma \tag{3}$$

[12] S. Silver, *Microwave antenna theory and design*, McGraw-Hill Book Co., 1949, p. 149.

[13] J. A. Stratton, *Electromagnetic theory*, McGraw-Hill Book Co.,1941, pp. 464-470.

where $k = 2\pi/\lambda$, E_0 is the amplitude of the incident electric field, η is the impedance of the medium in the upper half-space, and ι is the angle of incidence, as shown in Fig. 1.

Let $\mathbf{I}$ denote the stochastic integral in (1):

$$\mathbf{I} = \int_\Sigma \{\mathbf{n} \times \mathbf{H}_0 - [(\mathbf{n} \times \mathbf{H}_0)\cdot\mathbf{R}]\mathbf{R}\} \exp(ik\mathbf{d}\cdot\mathbf{R})\, dS, \tag{4}$$

and affix a subscript v or h to $\mathbf{I}$ according as the polarization is vertical or horizontal. The existence of the stochastic integral follows immediately from a theorem of Doob[14].

Now introduce (2) or (3) and the formulas

$$dS = (EG - F^2)^{1/2}\, dx\, dy;$$

$$\mathbf{n} = \left(-\frac{\partial Z}{\partial x}\mathbf{i} - \frac{\partial Z}{\partial y}\mathbf{j} + \mathbf{k}\right) \Big/ (EG - F^2)^{1/2}; \tag{5}$$

$$\mathbf{R} = \sin\alpha\mathbf{i} + \cos\alpha\mathbf{k}; \tag{6}$$

$$\mathbf{d} = x\mathbf{i} + y\mathbf{j} + Z(x, y)\mathbf{k}, \tag{7}$$

into (4), and let[15]

$$A(\alpha) = \sin\iota + \sin\alpha; \tag{8}$$

$$B(\alpha) = \cos\iota + \cos\alpha.$$

Then, apart from the time factor $\exp(i\omega t)$,

$$\mathbf{I}_v(\alpha) = (E_0/\eta)(-\mathbf{i}\cos\alpha + \mathbf{k}\sin\alpha)J(\alpha;\alpha); \tag{9}$$

$$\mathbf{I}_h(\alpha) = (E_0/\eta)\left\{\sin(\iota - \alpha)(\mathbf{i}\cos\alpha - \mathbf{k}\sin\alpha) \cdot \iint_D \frac{\partial Z}{\partial y} \exp(ik[Ax + BZ(x, y)])\, dx\, dy + \mathbf{j}J(\alpha;\iota)\right\}, \tag{10}$$

where

$$J(\alpha;\beta) = \iint_D \exp\{ik[A(\alpha)x + B(\alpha)Z(x, y)]\}\left(\cos\beta - \frac{\partial Z}{\partial x}\sin\beta\right) dx\, dy. \tag{11}$$

The energy flow at the field point P is given by the Poynting vector $\mathbf{S}(P)$:

$$\mathbf{S}(P) = \tfrac{1}{2}\eta^{-1}(\mathbf{E}_s\cdot\mathbf{E}_s^*)\mathbf{R} = \tfrac{1}{2}\eta^{-1}\omega^2\mu^2(2\pi R)^{-2}\mathbf{I}(\alpha)\cdot\mathbf{I}^*(\alpha)\mathbf{R}. \tag{12}$$

We thus have from (9) and (10) as the energy flows corresponding to each polarization:

$$\mathbf{S}_v(P) = \tfrac{1}{2}\eta^{-1}k^2(2\pi R)^{-2}E_0^2 J(\alpha;\alpha)J^*(\alpha;\alpha)\mathbf{R}; \tag{13}$$

$$\mathbf{S}_h(P) = \frac{1}{2}\eta^{-1}k^2E_0^2(2\pi R)^{-2}\left\{\sin^2(\iota - \alpha)\iiiint_{D\,D} \exp\{ik[A(x - x') + B(Z(x, y) - Z(x', y'))]\} \frac{\partial Z}{\partial y}\frac{\partial Z}{\partial y'}\, dx\, dy\, dx'\, dy' + J(\alpha;\iota)J^*(\alpha;\iota)\right\}\mathbf{R}. \tag{14}$$

[14] J. L. Doob, *Stochastic processes*, Wiley, 1953, theorem 2.7, p. 62. The existence is also a consequence[10] of the integrability of the covariance function of the integrand over D.

[15] It is implied by the form of (6) and (8) that the expressions developed below are restricted to field points P in the plane of incidence.

4. Mean and covariance of the scattered field for vertical polarization. Referring to expressions (3.1), (3.4), and (3.9) one sees that the expected value of the scattered field is proportional to $\mathcal{E}\{J(\alpha;\alpha)\}$ for vertically-polarized incident radiation. By Fubini's theorem and the representation (2.3), we have

$$\begin{aligned}\mathcal{E}\{J(\alpha;\alpha)\} &= \cos\alpha \iint_D \exp(ikAx)\mathcal{E}\{\exp[ikBZ(x,y)]\}\,dx\,dy \\ &\quad - \sin\alpha \iint_D \exp(ikAx)\,\mathcal{E}\Big\{\sum_{m,n}\lambda_{mn}^{-1/2}\frac{\partial\varphi_{mn}}{\partial x}z_{mn} \\ &\quad \cdot \exp\Big[ikB\sum_{p,q}\lambda_{pq}^{-1/2}\varphi_{pq}(x,y)z_{pq}\Big]\Big\}\,dx\,dy.\end{aligned} \tag{1}$$

In view of (2.8) and (2.9) this expression becomes

$$\begin{aligned}\mathcal{E}\{J(\alpha;\alpha)\} &= \cos\alpha \iint_D \exp(ikAx)\chi(kB;x,y)\,dx\,dy \\ &\quad + i\sin\alpha \iint_D \exp(ikAx)\sum_{m,n}\lambda_{mn}^{-1/2}\frac{\partial\varphi_{mn}}{\partial x}\frac{d}{dB_{mn}}\Big(\prod_{p,q}\chi(B_{pq})\Big)\,dx\,dy.\end{aligned} \tag{2}$$

If the surface is Gaussian, so that formulas (2.13) and (2.14) apply, the second expectation on the right hand side of (1) becomes

$$\begin{aligned}&\mathcal{E}\Big\{\sum_{m,n}\lambda_{mn}^{-1/2}\frac{\partial\varphi_{mn}}{\partial x}z_{mn}\exp\Big[ikB\sum_{p,q}\lambda_{pq}^{-1/2}\varphi_{pq}(x,y)z_{pq}\Big]\Big\} \\ &\qquad = -kB\exp\Big\{-\frac{1}{2}k^2B^2\sigma^2(x,y)\Big\}\cdot\frac{1}{2}\frac{\partial}{\partial x}\sum_{m,n}\lambda_{mn}^{-1}\varphi_{mn}^2(x,y) \\ &\qquad = -\frac{1}{2}kB\frac{\partial\sigma^2(x,y)}{\partial x}\exp\Big\{-\frac{1}{2}k^2B^2\sigma^2(x,y)\Big\}.\end{aligned} \tag{3}$$

Thus (1) becomes in the Gaussian case,

$$\mathcal{E}\{J(\alpha;\alpha)\} = \iint_D \exp\Big\{ikAx - \frac{1}{2}k^2B^2\sigma^2(x,y)\Big\}\Big[\cos\alpha - \frac{1}{2}ikB\sin\alpha\frac{\partial\sigma^2(x,y)}{\partial x}\Big]\,dx\,dy, \tag{4}$$

and if further the Z-process is stationary, so that σ^2 is constant, and if $D = (-a,a)\times(-b,b)$, then the expression for the mean of $J(\alpha;\alpha)$ reduces to

$$\mathcal{E}\{J(\alpha;\alpha)\} = 4ab\cos\alpha\exp\Big(-\frac{1}{2}k^2B^2\sigma^2\Big)\frac{\sin(kAa)}{kAa}. \tag{5}$$

We thus obtain the mean scattered field for vertical polarization in the form

$$\mathcal{E}\{\mathbf{E}_s(P)\} = -i\omega\mu(2\pi R)^{-1}\exp(-ikR)(-\mathbf{i}\cos\alpha + \mathbf{k}\sin\alpha)\mathcal{E}\{J(\alpha;\alpha)\}, \tag{6}$$

with $\mathcal{E}\{J(\alpha;\alpha)\}$ being given by (2), (4), or (5) according as the process specifying the random surface is separable, Gaussian, or stationary Gaussian with a rectangular parameter-domain D.

We now pass to a consideration of the covariance function of the vertically-polarized

scattered field, which according to (3.1) and (3.9), is related to the following expected value:

$$\begin{aligned}\mathcal{E}\{J(\alpha_1\,;\alpha_1)J^*(\alpha_2\,;\alpha_2)\} &= \iiiint_{D\ D} \exp\{ik(A_1x - A_2x')\} \\ &\cdot \Big(\cos\alpha_1\cos\alpha_2\,\mathcal{E}\{\exp[ik(B_1Z(x,y) - B_2Z(x',y'))]\} \\ &- \sin\alpha_1\cos\alpha_2\,\mathcal{E}\Big\{\frac{\partial Z}{\partial x}\exp[ik(B_1Z(x,y) - B_2Z(x',y'))]\Big\} \\ &- \sin\alpha_2\cos\alpha_1\,\mathcal{E}\Big\{\frac{\partial Z}{\partial x'}\exp[ik(B_1Z(x,y) - B_2Z(x',y'))]\Big\} \\ &+ \sin\alpha_1\sin\alpha_2\,\mathcal{E}\Big\{\frac{\partial Z}{\partial x}\frac{\partial Z}{\partial x'}\exp[ik(B_1Z(x,y) - B_2Z(x',y'))]\Big\}\Big)\,dx\,dy\,dx'\,dy'.\end{aligned} \tag{7}$$

Introducing the representation (2.3) and making use of formulas (2.10)-(2.12) enables us to write (7) in the case of a separable process as

$$\begin{aligned}\mathcal{E}\{J(\alpha_1\,;\alpha_1)J^*(\alpha_2\,;\alpha_2)\} &= \iiiint_{D\ D} \exp\{ik(A_1x - A_2x')\}\Big(\cos\alpha_1\cos\alpha_2\prod_{p,q}\chi(C_{pq}) \\ &+ i\sum_{m,n}\lambda_{mn}^{-1/2}\Big(\sin\alpha_1\cos\alpha_2\frac{\partial\varphi_{mn}}{\partial x} + \cos\alpha_1\sin\alpha_2\frac{\partial\varphi_{mn}}{\partial x'}\Big)\frac{d}{dC_{mn}}\Big(\prod_{p,q}\chi(C_{pq})\Big) \\ &- \sin\alpha_1\sin\alpha_2\Big\{\sum_{m,n}\lambda_{mn}^{-1}\frac{\partial\varphi_{mn}}{\partial x}\frac{\partial\varphi_{mn}}{\partial x'}\frac{d^2}{dC_{mn}^2}\Big(\prod_{p,q}\chi(C_{pq})\Big) + \sum_{m,n}\sum{}'(\lambda_{mn}\lambda_{pq})^{-1/2} \\ &\cdot \frac{\partial\varphi_{mn}}{\partial x}\frac{\partial\varphi_{pq}}{\partial x'}\frac{d\chi(C_{mn})}{dC_{mn}}\frac{d\chi(C_{pq})}{dC_{pq}}\prod{}''\chi(C_{st})\Big\}\Big)\,dx\,dy\,dx'\,dy',\end{aligned} \tag{8}$$

where Σ' indicates that the summation is to extend over $p \neq m$ and/or $q \neq n$, and Π'' has the same meaning as following Eq. (2.12).

If the process is Gaussian, we have from (2.3), (2.16), and (2.18) that

$$\begin{aligned}\mathcal{E}\Big(\frac{\partial Z}{\partial x}\exp\{ik[B_1Z(x,y) - B_2Z(x',y')]\}\Big) &= i\sum_{m,n}\lambda_{mn}^{-1/2}\frac{\partial\varphi_{mn}}{\partial x}C_{mn}\exp\Big(-\frac{1}{2}\sum_{p,q}C_{pq}^2\Big) \\ &= ik\exp\Big\{-\frac{1}{2}k^2Q(x,y;x',y')\Big\}\sum_{m,n}\lambda^{-1}\frac{\partial\varphi_{mn}}{\partial x}[B_1\varphi_{mn}(x,y) - B_2\varphi_{mn}(x',y')] \\ &= ik\exp\Big\{-\frac{1}{2}k^2Q(x,y;x',y')\Big\}\Big(\frac{1}{2}B_1\frac{\partial\sigma^2(x,y)}{\partial x} - B_2\frac{\partial r(x,y;x',y')}{\partial x}\Big);\end{aligned} \tag{9}$$

and similarly, from (2.3), (2.16), and (2.18),

$$\begin{aligned}\mathcal{E}\Big(\frac{\partial Z}{\partial x'}\exp\{ik[B_1Z(x,y) - B_2Z(x',y')]\}\Big) &= ik\exp\Big(-\frac{1}{2}k^2Q\Big)\Big(B_1\frac{\partial r(x,y;x',y')}{\partial x'} \\ &\quad - \frac{1}{2}B_2\frac{\partial\sigma^2(x',y')}{\partial x'}\Big).\end{aligned} \tag{10}$$

By means of (2.3), (2.17), and (2.18) we find also that for a Gaussian surface,

$$\mathcal{E}\left(\frac{\partial Z}{\partial x}\frac{\partial Z}{\partial x'}\exp\{ik[B_1Z(x,y)-B_2Z(x',y')]\}\right)=\exp\left(-\frac{1}{2}k^2Q\right)\Bigg\{\sum_{m,n}\lambda_{mn}^{-1}\frac{\partial\varphi_{mn}}{\partial x}\frac{\partial\varphi_{mn}}{\partial x'}$$
$$-k^2\Bigg[B_1^2\sum_{m,n}\sum_{p,q}\lambda_{mn}^{-1}\frac{\partial\varphi_{mn}}{\partial x}\varphi_{mn}(x,y)\lambda_{pq}^{-1}\frac{\partial\varphi_{pq}}{\partial x'}\varphi_{pq}(x,y)+B_2^2\sum_{m,n}\sum_{p,q}\lambda_{mn}^{-1}\frac{\partial\varphi_{mn}}{\partial x}\varphi_{mn}(x',y')$$
$$\cdot\lambda_{pq}^{-1}\frac{\partial\varphi_{pq}}{\partial x'}\varphi_{pq}(x',y')-B_1B_2\sum_{m,n}\sum_{p,q}(\lambda_{mn}\lambda_{pq})^{-1}\Bigg(\frac{\partial\varphi_{mn}}{\partial x}\varphi_{mn}(x,y)\frac{\partial\varphi_{pq}}{\partial x'}\varphi_{pq}(x',y')$$
$$+\frac{\partial\varphi_{mn}}{\partial x}\varphi_{mn}(x',y')\frac{\partial\varphi_{pq}}{\partial x'}\varphi_{pq}(x,y)\Bigg)\Bigg]\Bigg\}$$
$$=\exp\left(-\frac{1}{2}k^2Q\right)\Bigg\{\frac{\partial^2r(x,y;x',y')}{\partial x\,\partial x'}-k^2\Bigg[\frac{1}{2}B_1^2\frac{\partial\sigma^2(x,y)}{\partial x}\frac{\partial r}{\partial x'}$$
$$+\frac{1}{2}B_2^2\frac{\partial r}{\partial x}\frac{\partial\sigma^2(x',y')}{\partial x'}-\frac{1}{4}B_1B_2\frac{\partial\sigma^2(x,y)}{\partial x}\frac{\partial\sigma^2(x',y')}{\partial x'}-B_1B_2\frac{\partial r}{\partial x}\frac{\partial r}{\partial x'}\Bigg]\Bigg\}. \tag{11}$$

Thus (7) becomes, for a Gaussian process,

$$\mathcal{E}\{J(\alpha_1\,;\alpha_1)J^*(\alpha_2\,;\alpha_2)\}=\iiiint_{D\;D}\exp\Bigg\{ik(A_1x-A_2x')-\frac{1}{2}k^2[B_1^2\sigma^2(x,y)+B_2^2\sigma^2(x',y')$$
$$-2B_1B_2\,r(x,y;x',y')]\Bigg\}\Bigg\{\cos\alpha_1\cos\alpha_2+\sin\alpha_1\sin\alpha_2\frac{\partial^2r(x,y;x',y')}{\partial x\,\partial x'}$$
$$-ik\sin\alpha_1\cos\alpha_2\Bigg[\frac{1}{2}B_1\frac{\partial\sigma^2(x,y)}{\partial x}-B_2\frac{\partial r}{\partial x}\Bigg]-ik\cos\alpha_1\sin\alpha_2\Bigg[B_1\frac{\partial r}{\partial x'}-\frac{1}{2}B_2\frac{\partial\sigma^2}{\partial x'}\Bigg]$$
$$-k^2\sin\alpha_1\sin\alpha_2\Bigg[\frac{1}{2}B_1^2\frac{\partial\sigma^2(x,y)}{\partial x}\frac{\partial r}{\partial x'}+\frac{1}{2}B_2^2\frac{\partial r}{\partial x}\frac{\partial\sigma^2(x',y')}{\partial x'}-\frac{1}{4}B_1B_2\frac{\partial\sigma^2(x,y)}{\partial x}\frac{\partial\sigma^2(x',y')}{\partial x'}$$
$$-B_1B_2\frac{\partial r}{\partial x}\frac{\partial r}{\partial x'}\Bigg]\Bigg\}\,dx\,dy\,dx'\,dy'. \tag{12}$$

Suppose now that the process is stationary as well as Gaussian. Then the variance $\sigma^2(x,y)$ is constant, and letting u and v denote the displacements $x'-x$ and $y'-y$, respectively, the covariance function of the surface becomes

$$r(x,y;x',y')=r(u,v); \tag{13}$$

so that

$$\partial r/\partial x'=\partial r/\partial u=-\partial r/\partial x; \tag{14}$$
$$\partial^2r/\partial x\,\partial x'=-\partial^2r/\partial u^2. \tag{15}$$

Thus (12) becomes for the stationary Gaussian case,

$$\mathcal{E}\{J(\alpha_1\,;\alpha_1)J^*(\alpha_2\,;\alpha_2)\}=\exp\{-\tfrac{1}{2}k^2\sigma^2(B_1^2+B_2^2)\}$$
$$\cdot\iiiint_{D\;D'}\exp\{ik(A_1-A_2)x-ikA_2u+k^2B_1B_2r(u,v)\}\Bigg\{\cos\alpha_1\cos\alpha_2$$
$$-\sin\alpha_1\sin\alpha_2\frac{\partial^2r}{\partial u^2}-ik\frac{\partial r}{\partial u}(B_2\sin\alpha_1\cos\alpha_2+B_1\cos\alpha_1\sin\alpha_2)$$
$$-k^2B_1B_2\sin\alpha_1\sin\alpha_2\left(\frac{\partial r}{\partial u}\right)^2\Bigg\}\,dx\,dy\,du\,dv.$$

If as in (5), $D = (-a, a) \times (-b, b)$, then this expression may be simplified. First write the integral as

$$\int_{-a}^{a}\int_{-a-x}^{a-x}\int_{-b}^{b}\int_{-b-y}^{b-y} \exp\,[icx - ikA_2u][g(u, v) - ih(u, v)]\,dx\,du\,dy\,dv, \tag{16}$$

where

$$c = k(A_1 - A_2), \tag{17}$$

$$g(u, v) = \exp\,[k^2B_1B_2r(u, v)]\left[\cos\alpha_1\cos\alpha_2 - \sin\alpha_1\sin\alpha_2\frac{\partial^2 r}{\partial u^2} - k^2B_1B_2\sin\alpha_1\sin\alpha_2\left(\frac{\partial r}{\partial u}\right)^2\right], \tag{18}$$

$$h(u, v) = \exp\,[k^2B_1B_2r(u, v)]k(B_2\sin\alpha_1\cos\alpha_2 + B_1\cos\alpha_1\sin\alpha_2)\frac{\partial r}{\partial u}. \tag{19}$$

From the definition of covariance and the hypothesis of stationarity, we have immediately that

$$r(-u, -v) = r(u, v) \qquad \text{and} \qquad r(-u, v) = r(u, -v). \tag{20}$$

It follows that

$$\frac{\partial r(-u, -v)}{\partial u} = -\frac{\partial r(u, v)}{\partial u}; \qquad -\frac{\partial r(-u, v)}{\partial u} = \frac{\partial r(u, -v)}{\partial u}; \tag{21}$$

$$\frac{\partial r(-u, -v)}{\partial v} = -\frac{\partial r(u, v)}{\partial v}; \qquad -\frac{\partial r(-u, v)}{\partial v} = \frac{\partial r(u, -v)}{\partial v}; \tag{22}$$

and

$$\frac{\partial^2 r(-u, -v)}{\partial u^2} = \frac{\partial^2 r(u, v)}{\partial u^2}; \qquad \frac{\partial^2 r(-u, v)}{\partial u^2} = \frac{\partial^2 r(u, -v)}{\partial u^2}; \tag{23}$$

$$\frac{\partial^2 r(-u, -v)}{\partial v^2} = \frac{\partial^2 r(u, v)}{\partial v^2}; \qquad \frac{\partial^2 r(-u, v)}{\partial v^2} = \frac{\partial^2 r(u, -v)}{\partial v^2}. \tag{24}$$

Consequently

$$g(-u, -v) = g(u, v); \qquad g(-u, v) = g(u, -v); \tag{25}$$

and

$$h(-u, -v) = -h(u, v); \qquad h(-u, v) = -h(u, -v). \tag{26}$$

Next invert the order of integration in (16) and perform the x,y-integration. The result is, after some obvious changes of variable and use is made of relations (25) and (26):

$$\frac{4}{c}\int_0^{2a}\int_0^{2b}\sin\left[c\left(a - \frac{u}{2}\right)\right](2b - v)\left\{\cos\left(\frac{1}{2}c'u\right)[g(u, v) + g(u, -v)] - \sin\left(\frac{1}{2}c'u\right)[h(u, v) + h(u, -v)]\right\}du\,dv,$$

where

$$c' = k(A_1 + A_2). \tag{27}$$

Thus, for a stationary Gaussian random surface with rectangular parameter-domain[16]:

$$\begin{aligned}\mathcal{E}\{J(\alpha_1\,;\alpha_1)J^*(\alpha_2\,;\alpha_2)\} &= 4k^{-1}(A_1 - A_2)^{-1}\exp\{-\tfrac{1}{2}k^2\sigma^2(B_1^2 + B_2^2)\}\\ &\cdot\int_0^{2a}\int_0^{2b}\sin\left[k(A_1 - A_2)\left(a - \frac{u}{2}\right)\right](2b - v)\\ &\cdot\{\cos[\tfrac{1}{2}k(A_1 + A_2)u][g(u, v) + g(u, -v)]\\ &- \sin[\tfrac{1}{2}k(A_1 + A_2)u][h(u, v) + h(u, -v)]\}\,du\,dv.\end{aligned} \tag{28}$$

The covariance of the scattered field in the vertically-polarized case is then given by

$$\begin{aligned}\operatorname{cov}[\mathbf{E}_s(P_1), \mathbf{E}_s(P_2)] &= \mathcal{E}\{[\mathbf{E}_s(P_1) - \mathcal{E}\mathbf{E}_s(P_1)]\cdot[\mathbf{E}_s(P_2) - \mathcal{E}\mathbf{E}_s(P_2)]^*\}\\ &= (2\pi)^{-2}k^2E_0^2(R_1R_2)^{-1}\cos(\alpha_1 - \alpha_2)\exp\{-ik(R_1 - R_2)\}\\ &\cdot[\mathcal{E}\{J(\alpha_1\,;\alpha_1)J^*(\alpha_2\,;\alpha_2)\} - \mathcal{E}\{J(\alpha_1\,;\alpha_1)\}\mathcal{E}^*\{J(\alpha_2\,;\alpha_2)\}],\end{aligned} \tag{29}$$

where $\mathcal{E}\{J(\alpha_1\,;\alpha_1)J^*(\alpha_2\,;\alpha_2)\}$ and $\mathcal{E}\{J(\alpha;\alpha)\}$ are given by (8) and (2), (12) and (4), or (28) and (5), according as the Z-process is separable, Gaussian, or stationary Gaussian with rectangular parameter-domain D.

The variance of the scattered field for vertical polarization follows at once from (29) by setting $P_1 = P_2 = P$ (whence $\alpha_1 = \alpha_2 = \alpha$). In the stationary Gaussian, rectangular parameter-domain case, for example, one obtains the following expression for the variance of the scattered field:

$$\begin{aligned}\operatorname{var}[\mathbf{E}_s(P)] &= \tfrac{1}{2}k^2E_0^2(\pi R)^{-2}\exp(-k^2\sigma^2B^2)\\ &\cdot\left(\int_0^{2a}\int_0^{2b}(2a - u)(2b - v)\{\cos(kAu)[g_a(u, v) + g_a(u, -v)]\right.\\ &\left.- \sin(kAu)[h_a(u, v) + h_a(u, -v)]\}\,du\,dv - 8a^2b^2\cos^2\alpha\,\frac{\sin^2(kAa)}{(kAa)^2}\right),\end{aligned} \tag{30}$$

where $g_a(u, v)$ and $h_a(u, v)$ are given by (18) and (19) upon setting $\alpha_1 = \alpha_2 = \alpha$.

The mean square energy flow, specified by the Poynting vector (3.13), is intimately connected with the variance (30). The mean square energy flow at P is obtained by setting $P_1 = P_2 = P$ in (8), (12), or (28) according as the Z-process is separable, Gaussian, or stationary Gaussian with rectangular domain D, and multiplying the resulting expression by $1/2\eta^{-1}k^2E_0^2(2\pi R)^{-2}\mathbf{R}$.

5. Mean and covariance of the scattered field for horizontal polarization. Examination of Eq. (3.10) shows that the expressions (4.2)-(4.5) already developed can be used to determine that portion of $\mathcal{E}\{\mathbf{I}_h(\alpha)\}$ involving the expectation of $J(\alpha;\iota)$. The remaining part of $\mathcal{E}\{\mathbf{I}_h(\alpha)\}$ can be evaluated in exactly the same fashion as for the second term of $\mathcal{E}\{J(\alpha;\alpha)\}$, i.e., as in (4.1). The result, for a separable random surface $Z(x, y)$, is

$$\begin{aligned}\mathcal{E}\{\mathbf{I}_h(\alpha)\} &= (E_0/\eta)\iint_D\exp(ikAx)\left\{\mathbf{j}\cos\iota\,\chi(kB;\,x,\,y)\right.\\ &+ i\sum_{m,n}\lambda_{mn}^{-1/2}\frac{d}{dB_{mn}}\Big(\prod_{p,q}\chi(B_{pq})\Big)\left[\mathbf{j}\sin\iota\,\frac{\partial\varphi_{mn}}{\partial x}\right.\\ &\left.\left.- (\mathbf{i}\cos\alpha - \mathbf{k}\sin\alpha)\sin(\iota - \alpha)\,\frac{\partial\varphi_{mn}}{\partial y}\right]\right\}dx\,dy.\end{aligned} \tag{1}$$

[16]The author is indebted to the referee for pointing out the proper form of this expression.

If the surface is Gaussian, it follows from (2.13) and (2.14) that

$$\mathcal{E}\{\mathbf{I}_h(\alpha)\} = (E_0/\eta) \iint_D \exp\left\{ikAx - \frac{1}{2}k^2B^2\sigma^2(x, y)\right\}\Big\{\mathbf{j}\cos\iota + \frac{1}{2}ikB\Big[(\mathbf{i}\cos\alpha - \mathbf{k}\sin\alpha)\sin(\iota - \alpha)\frac{\partial\sigma^2(x, y)}{\partial y} - \mathbf{j}\sin\iota\frac{\partial\sigma^2(x, y)}{\partial x}\Big]\Big\}\,dx\,dy. \tag{2}$$

If the surface is stationary as well as Gaussian, and if $D = (-a, a) \times (-b, b)$, then (2) reduces to

$$\mathcal{E}\{\mathbf{I}_h(\alpha)\} = \mathbf{j}4ab\cos\iota(E_0/\eta)\exp\left(-\frac{1}{2}k^2B^2\sigma^2\right)\frac{\sin(kAa)}{(kAa)}. \tag{3}$$

The mean scattered field for horizontal polarization is thus given by the expression:

$$\mathcal{E}\{\mathbf{E}_s(P)\} = -\frac{i\omega\mu}{2\pi R}\exp(-ikR)\mathcal{E}\{\mathbf{I}_h(\alpha)\}, \tag{4}$$

with $\mathcal{E}\{\mathbf{I}_h(\alpha)\}$ from (1), (2), or (3) according as the process $Z(x, y)$ representing the surface is separable, Gaussian, or stationary and Gaussian with rectangular domain D.

Next the covariance of the scattered field will be determined for horizontal polarization. From (3.10),

$$\mathcal{E}\{\mathbf{I}_h(\alpha_1)\cdot\mathbf{I}_h^*(\alpha_2)\} = (E_0/\eta)^2\Big[\sin(\iota - \alpha_1)\sin(\iota - \alpha_2)\cos(\alpha_1 - \alpha_2) \cdot \iint_D\iint_D \exp\{ik(A_1x - A_2x')\}\mathcal{E}\Big\{\frac{\partial Z}{\partial y}\frac{\partial Z}{\partial y'}\exp(ik[B_1Z(x, y) - B_2Z(x', y')])\Big\}\,dx\,dy\,dx'\,dy' + \mathcal{E}\{J(\alpha_1\,;\,\iota)J^*(\alpha_2\,;\,\iota)\}\Big]. \tag{5}$$

In the subsequent computations the expressions (4.8), (4.12), or (4.28) already developed can be used to find $\mathcal{E}\{J(\alpha_1\,;\,\iota)J^*(\alpha_2\,;\,\iota)\}$. From (2.12) we have that

$$\begin{aligned}\mathcal{E}\Big\{\frac{\partial Z}{\partial y}\frac{\partial Z}{\partial y'}\exp(ik[B_1Z(x, y) - B_2Z(x', y')])\Big\} &= \sum_{m,n}\sum_{p,q}(\lambda_{mn}\lambda_{pq})^{-1/2}\frac{\partial\varphi_{mn}}{\partial y}\frac{\partial\varphi_{pq}}{\partial y'} \cdot \mathcal{E}\{z_{mn}z_{pq}\exp(i\sum_{s,t}C_{st}z_{st})\} \\ &= -\sum_{m,n}\lambda_{mn}^{-1}\frac{\partial\varphi_{mn}}{\partial y}\frac{\partial\varphi_{mn}}{\partial y'}\frac{d^2}{dC_{mn}^2}\Big[\prod_{p,q}\chi(C_{pq})\Big] \\ &\quad - \sum_{m,n}\sum{}'(\lambda_{mn}\lambda_{pq})^{-1/2}\frac{\partial\varphi_{mn}}{\partial y}\frac{\partial\varphi_{pq}}{\partial y'}\frac{d\chi(C_{mn})}{dC_{mn}}\frac{d\chi(C_{pq})}{dC_{pq}}\prod{}''\chi(C_{st}),\end{aligned} \tag{6}$$

where Σ' and Π'' have the same meanings as following expression (2.12). Substituting this expression into (5) and modifying (4.8) as required for $\mathcal{E}\{J(\alpha_1\ ;\ \iota)J^*(\alpha_2\ ;\ \iota)\}$, one obtains in the case of a separable process:

$$\begin{aligned}
\mathcal{E}\{\mathbf{I}_h(\alpha_1)\cdot\mathbf{I}_h^*(\alpha_2)\} &= (E_0/\eta)^2\Big(-\sin(\iota-\alpha_1)\sin(\iota-\alpha_2)\cos(\alpha_1-\alpha_2)\\
&\cdot\iint_D\iint_D \exp\{ik(A_1x-A_2x')\}\Big\{\sum_{m,n}\lambda_{mn}^{-1}\frac{\partial\varphi_{mn}}{\partial y}\frac{\partial\varphi_{mn}}{\partial y'}\frac{d^2}{dC_{mn}^2}\Big[\prod_{p,q}\chi(C_{pq})\Big]\\
&+\sum_{m,n}\sum{}'(\lambda_{mn}\lambda_{pq})^{-1/2}\frac{\partial\varphi_{mn}}{\partial y}\frac{\partial\varphi_{pq}}{\partial y'}\frac{d\chi(C_{mn})}{dC_{mn}}\frac{d\chi(C_{pq})}{dC_{pq}}\prod{}''\chi(C_{s\iota})\Big\}\,dx\,dy\,dx'\,dy'\\
&+\iint_D\iint_D \exp\{ik(A_1x-A_2x')\}\Big\{\cos^2\iota\prod_{p,q}\chi(C_{pq})+i\sum_{m,n}\lambda_{mn}^{-1/2}\sin\iota\cos\iota\\
&\cdot\Big(\frac{\partial\varphi_{mn}}{\partial x}+\frac{\partial\varphi_{mn}}{\partial x'}\Big)\frac{d}{dC_{mn}}\Big[\prod_{p,q}\chi(C_{pq})\Big]-\sin^2\iota\Big(\sum_{m,n}\lambda_{mn}^{-1}\frac{\partial\varphi_{mn}}{\partial x}\frac{\partial\varphi_{mn}}{\partial x'}\frac{d^2}{dC_{mn}^2}\Big[\prod_{p,q}\chi(C_{pq})\Big]\\
&+\sum_{m,n}\sum{}'(\lambda_{mn}\lambda_{pq})^{-1/2}\frac{\partial\varphi_{mn}}{\partial x}\frac{\partial\varphi_{pq}}{\partial x'}\frac{d\chi(C_{mn})}{dC_{mn}}\frac{d\chi(C_{pq})}{dC_{pq}}\prod{}''\chi(C_{s\iota})\Big)\Big\}\,dx\,dy\,dx'\,dy'\Big).
\end{aligned}\tag{7}$$

If the process representing the surface is Gaussian, Eq. (6) reduces to an expression like (4.11), the only difference being that the partial differentiations are to be performed with respect to y and y' rather than x and x'. Thus (5) becomes, for a Gaussian surface,

$$\begin{aligned}
\mathcal{E}\{\mathbf{I}_h(\alpha_1)\cdot\mathbf{I}_h^*(\alpha_2)\} &= (E_0/\eta)^2\iint_D\iint_D \exp\Big\{ik(A_1x-A_2x')\\
&-\frac{1}{2}k^2[B_1^2\sigma^2(x,y)+B_2^2\sigma^2(x',y')-2B_1B_2r(x,y;x',y')]\Big\}\\
&\cdot\Big\{\sin(\iota-\alpha_1)\sin(\iota-\alpha_2)\cos(\alpha_1-\alpha_2)\Big(\frac{\partial^2 r}{\partial y\,\partial y'}-k^2\Big[\frac{1}{2}B_1^2\frac{\partial\sigma^2(x,y)}{\partial y}\frac{\partial r}{\partial y'}\\
&+\frac{1}{2}B_2^2\frac{\partial r}{\partial y}\frac{\partial\sigma^2(x',y')}{\partial y'}-\frac{1}{4}B_1B_2\frac{\partial\sigma^2(x,y)}{\partial y}\frac{\partial\sigma^2(x',y')}{\partial y'}-B_1B_2\frac{\partial r}{\partial y}\frac{\partial r}{\partial y'}\Big]\Big)\\
&+\cos^2\iota+\sin^2\iota\frac{\partial^2 r}{\partial x\,\partial x'}-ik\sin\iota\cos\iota\Big[\frac{1}{2}\Big(B_1\frac{\partial\sigma^2}{\partial x}-B_2\frac{\partial\sigma^2}{\partial x'}\Big)\\
&+\Big(B_1\frac{\partial r}{\partial x'}-B_2\frac{\partial r}{\partial x}\Big)\Big]-k^2\sin^2\iota\Big[\frac{1}{2}\Big(B_1^2\frac{\partial\sigma^2}{\partial x}\frac{\partial r}{\partial x'}+B_2^2\frac{\partial r}{\partial x}\frac{\partial\sigma^2}{\partial x'}\Big)\\
&-B_1B_2\Big(\frac{\partial r}{\partial x}\frac{\partial r}{\partial x'}+\frac{1}{4}\frac{\partial\sigma^2}{\partial x}\frac{\partial\sigma^2}{\partial x'}\Big)\Big]\Big\}\,dx\,dy\,dx'\,dy'.
\end{aligned}\tag{8}$$

Suppose now that the surface is stationary as well as Gaussian and that the parameter-domain is rectangular: $D = (-a, a) \times (-b, b)$. Since $u = x' - x$ and $v = y' - y$,

we have in addition to formulas (4.14) and (4.15) that

$$\partial r/\partial y' = \partial r/\partial v = -\partial r/\partial y; \tag{9}$$

and

$$\partial^2 r/\partial y\, \partial y' = -\partial^2 r/\partial v^2. \tag{10}$$

On account of the stationarity, $\sigma^2(x, y) = \text{const.}$, and (8) can be written as

$$\begin{aligned}\mathcal{E}\{\mathbf{I}_h(\alpha_1)\cdot\mathbf{I}_h^*(\alpha_2)\} &= (E_0/\eta)^2 \exp\{-\tfrac{1}{2}k^2\sigma^2(B_1^2 + B_2^2)\} \\ &\cdot \int_{-a}^{a} dx \int_{-b}^{b} dy \int_{-a-x}^{a-x} du \int_{-b-y}^{b-y} dv \exp(icx - ikA_2u)[G(u, v) - iH(u, v)],\end{aligned} \tag{11}$$

where c is given by (4.17), and

$$\begin{aligned}G(u, v) = \exp[k^2B_1B_2r(u, v)]\Bigg\{&\cos^2\iota - \sin^2\iota\,\frac{\partial^2 r}{\partial u^2} - k^2\sin^2\iota\, B_1B_2\left(\frac{\partial r}{\partial u}\right)^2 \\ &- \sin(\iota - \alpha_1)\sin(\iota - \alpha_2)\cos(\alpha_1 - \alpha_2)\left[\frac{\partial^2 r}{\partial v^2} + k^2B_1B_2\left(\frac{\partial r}{\partial v}\right)^2\right]\Bigg\};\end{aligned} \tag{12}$$

$$H(u, v) = \exp[k^2B_1B_2r(u, v)]k\sin\iota\cos\iota\,(B_1 + B_2)\frac{\partial r}{\partial u}. \tag{13}$$

It follows from expressions (4.21) to (4.24) that

$$G(-u, -v) = G(u, v); \qquad G(-u, v) = G(u, -v); \tag{14}$$

$$H(-u, -v) = -H(u, v); \qquad H(-u, v) = -H(u, -v). \tag{15}$$

Formally, then, the integral in (11) resembles that in Eq. (4.16), and we can therefore write (11) at once in the form

$$\begin{aligned}\mathcal{E}\{\mathbf{I}_h(\alpha_1)\cdot\mathbf{I}_h^*(\alpha_2)\} &= 4k^{-1}(A_1 - A_2)^{-1}(E_0/\eta)^2 \exp\{-\tfrac{1}{2}k^2\sigma^2(B_1^2 + B_2^2)\} \\ &\cdot \int_0^{2a}\int_0^{2b} \sin\left[k(A_1 - A_2)\left(a - \frac{u}{2}\right)\right](2b - v)\{\cos[\tfrac{1}{2}k(A_1 + A_2)u] \\ &\cdot [G(u, v) + G(u, -v)] - \sin[\tfrac{1}{2}k(A_1 + A_2)u][H(u, v) + H(u, -v)]\}\, du\, dv.\end{aligned} \tag{16}$$

Then the covariance of the scattered field for the case of horizontal polarization is given by

$$\begin{aligned}\operatorname{cov}[\mathbf{E}_s(P_1), \mathbf{E}_s(P_2)] &= (2\pi)^{-2}\omega^2\mu^2(R_1R_2)^{-1}\exp\{-ik(R_1 - R_2)\} \\ &\cdot [\mathcal{E}\{\mathbf{I}_h(\alpha_1)\cdot\mathbf{I}_h^*(\alpha_2)\} - \mathcal{E}\{\mathbf{I}_h(\alpha_1)\}\cdot\mathcal{E}^*\{\mathbf{I}_h(\alpha_2)\}],\end{aligned} \tag{17}$$

where the terms in the square bracket on the right are to be supplied from (7) and (1), (8) and (2), or (16) and (3) according as the process representing the random surface is separable, Gaussian, or stationary and Gaussian with rectangular domain D.

The variance of the scattered field for the case of horizontal polarization is obtained from (17) by setting $P_1 = P_2 = P$, so that $\alpha_1 = \alpha_2 = \alpha$. For instance in the stationary Gaussian case, for a rectangular parameter-domain D, one obtains in this way the

following expression for the variance of the scattered field:

$$\text{var } [\mathbf{E}_s(P)] = \tfrac{1}{2}k^2E_0^2(\pi R)^{-2} \exp(-k^2\sigma^2B^2)\Big(\int_0^{2a}\int_0^{2b}(2a-u)(2b-v)$$
$$\cdot \{\cos(kAu)[G_a(u,v)+G_a(u,-v)]$$
$$-\sin(kAu)[H_a(u,v)+H_a(u,-v)]\}\,du\,dv - 8a^2b^2\cos^2\iota\,\frac{\sin^2(kAa)}{(kAa)^2}\Big). \tag{18}$$

Expressions (7), (8), and (16) are of course closely related to the mean square energy flow $\mathcal{E}\{\mathbf{S}_h(P)\}$ at the field point P. As indicated by Eq. (3.14) one sets $P_1 = P_2 = P$ in (7), (8), or (16), and then multiplies the resulting expression by $1/2\eta^{-1}\omega^2\mu^2(2\pi R)^{-2}\mathbf{R}$ in order to obtain the mean value of the Poynting vector $\mathbf{S}_h(P)$.

—NOTES—

SUBHARMONIC OSCILLATIONS IN A NONLINEAR SYSTEM WITH POSITIVE DAMPING*

By STIG LUNDQUIST (*The Royal Institute of Technology, Stockholm*)

Introduction. It is known from general theory[1,2] as well as from experiments, that the oscillations in a system with nonlinear restoring force, linear positive damping and an external driving force of period $2\pi/p$ tend to become periodic of period $k2\pi/p$ (k an integer), harmonic ($k = 1$) or subharmonic ($k > 1$). Subharmonic oscillations often have a much larger amplitude than the corresponding harmonic oscillations, and this is important in many applications, e.g. in electric circuits containing iron. Much work has been done on subharmonics[3], but the problem cannot be considered as solved. In most theories a Fourier series approximation with a few terms is used. However, the convergence of the series may be slow, the method will work only for a polynomial form of the nonlinear characteristic, it is laborious even for small k, and the stability of the solution is not easy to verify.

In this paper subharmonic oscillations are discussed for a general form of the restoring force by means of a difference method due to Cartwright and Littlewood and applied by the former to the van der Pol equation with forcing term. The method is especially useful for high orders. Preliminary experiments agree with theoretical results.

Method. For the following reasons the oscillation is considered as a free oscillation, slightly disturbed by the damping, the driving force and the nonlinearity.

1) From experiments it is known that subharmonics exist only if the damping is small.
2) The physically interesting case is that of a large subharmonic instead of a small harmonic oscillation.
3) A small nonlinearity makes it possible to work with first approximations although it does not always correspond to observed physical situations. The discussion is therefore confined to the equation

$$y'' + y = \epsilon\{F \sin(px + a) - \kappa y' - f(y)\},$$

$$\epsilon, F, \kappa > 0; \qquad f(y) + f(-y) = 0, \tag{1}$$

$$\epsilon \ll 1; \qquad f(y)/y \geq 0.$$

The derivative df/dy is assumed to exist for all values of y (weaker conditions are possible). The equation therefore has a unique solution, which is a continuous function of x and of the initial values.

For $\epsilon = 0$, Eq. (1) has solutions $b \sin x$ with b arbitrary. The solution for $\epsilon \neq 0$ can be expected to be close to one of these solutions in a finite interval[4]. The introduction

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[1] N. Levinson, Ann. Math. 45, 723, 1944.

[2] M. Cartwright, Ann. Math. Studies 20, Princeton, 1950.

[3] Ref. in Stoker, *Nonlinear vibrations in mechanical and electrical systems*, New York, 1950.

[4] H. Poincaré, *Les méthodes nouvelles de la mécanique céleste*, Vol. 1, Paris, 1892.

of the phase α makes it possible to write

$$y = b \sin x + \epsilon\eta(x) \tag{2}$$

A solution with initial values

$$y(0) = y'(0) = 0 \tag{3}$$

will cross the x-axis at $x = 2\pi + \tau$ where τ is small, with a value of the derivative $y'(2\pi + \tau) = b'$ and a phase

$$\alpha' = p(2\pi + \tau) + \alpha - k2\pi, \tag{4}$$

where k is the number of full periods of the driving force in the interval $0 \leq x \leq 2\pi + \tau$ (Fig. 1).

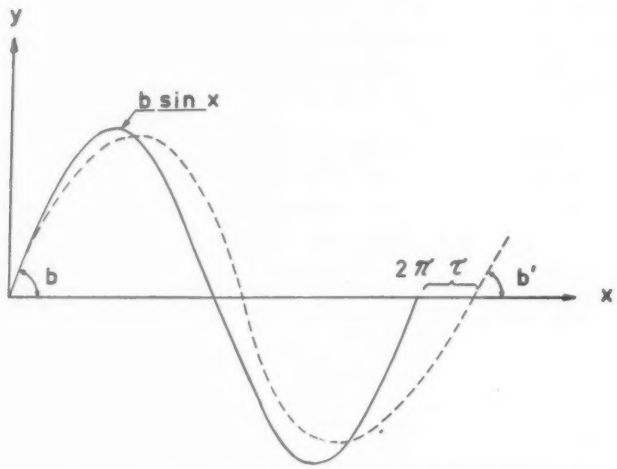

FIG. 1. Generating solution and disturbed solution (dotted)

Substituting b' for b and α' for α the process can be repeated. The solutions set up a transformation T of the point (b, α) on to $(b', \alpha') = T(b, \alpha)$. This transformation, which is (1,1) and continuous, is of interest only for $b > 0$, $0 \leq \alpha \leq 2\pi$.

The points fixed under T correspond to subharmonic solutions of period $k2\pi/p$ and the stability is determined by the properties of T in the neighborhood of a fixed point. For further information about this method the reader is referred to Reference 2.

Difference equations. The calculation of T is equivalent to finding a solution of (1) with initial values (3), valid for $0 \leq x \leq 2\pi + \tau$. This solution can be derived by successive approximations according to

$$y_{n+1}(x) = b \sin x + \epsilon \int_0^x \{F \sin (p\xi + \alpha) - \kappa y_n(\xi) - f(y_n)\} \sin (x - \xi)\, d\xi. \tag{5}$$

The functions y_n form a Cauchy-sequence converging for every finite x towards a continuous function y satisfying (1) and (3). The upper bound of $| y - y_n |$ is easily calculated. For ϵ small the first approximation is satisfactory.

Since τ is small of order ϵ

$$y(2\pi + \tau) = b\tau - \epsilon \int_0^{2\pi} \{F \sin(p\xi + \alpha) - \kappa b \cos\xi - f(b \sin\xi)\} \sin\xi \, d\xi + O(\epsilon^2) = 0, \tag{6}$$

$$y'(2\pi + \tau) = b + \epsilon \int_0^{2\pi} \{F \sin(p\xi + \alpha) - \kappa b \cos\xi - f(b \sin\xi)\} \cos\xi \, d\xi + O(\epsilon^2) = b, \tag{7}$$

and the difference equations defining T become

$$\frac{b' - b}{2\pi} = \frac{\epsilon F p}{\pi(p^2 - 1)} \sin p\pi \sin(p\pi + \alpha) - \frac{\epsilon\kappa b}{2} + O(\epsilon^2) = A(\alpha, b), \tag{8}$$

$$\frac{\alpha' - \alpha}{2\pi} = \frac{\epsilon F p}{\pi(p^2 - 1)} \sin p\pi \cos(p\pi + \alpha) - \epsilon p \frac{G(b)}{b} + p - k + O(\epsilon^2) = B(\alpha, b), \tag{9}$$

where

$$G(b) = \frac{1}{2\pi} \int_0^{2\pi} f(b \sin\xi) \sin\xi \, d\xi. \tag{10}$$

Periodic solutions. The subharmonic solutions of period $k\ 2\pi/p$ are found for $b' = b$, $\alpha' = \alpha$. If the curves $A = 0$, $B = 0$ are not nearly parallel the existence of an exactly periodic solution is obvious. To the first order it is given by

$$\frac{\epsilon F}{\pi b(p^2 - 1)} \sin p\pi \sin(p\pi + \alpha) = \frac{\epsilon\kappa}{2p}, \tag{11}$$

$$\frac{\epsilon F}{\pi b(p^2 - 1)} \sin p\pi \cos(p\pi + \alpha) = \frac{\epsilon G(b)}{b} - \frac{p - k}{p}. \tag{12}$$

These equations determine the amplitude and the phase. After squaring and adding one finds the amplitude equation

$$\left[\frac{\epsilon F \sin p\pi}{\pi b(p^2 - 1)}\right]^2 = \left[\frac{\epsilon\kappa}{2p}\right]^2 + \left[\frac{\epsilon G(b)}{b} - \frac{p - k}{p}\right]^2. \tag{13}$$

For a discussion of the amplitude as a function of the frequency it is convenient to write

$$\frac{\epsilon G(b)}{b} = \frac{p - k}{p} \pm C, \tag{14}$$

$$b = \frac{\epsilon F\, |\sin p\pi|}{\pi(p^2 - 1)} \left(\frac{\epsilon\kappa}{2p} + C^2\right)^{-1/2}. \tag{15}$$

The curves corresponding to (14) and (15) are easily constructed for different values of C and their points of intersection give the curves $b(p)$ (Fig. 2). In general these curves have closed parts near the curve defined by (14) for $C = 0$, and also a low branch, where the first approximation is not very good, however, since ϵF is not small compared to b. From the practical point of view, too, only the large amplitude subharmonic is of interest and so the low branch will not be discussed.

Eq. (13) is valid if $(p - k) \ll p$, which means a large interval of frequency for $k \gg 1$. The method permits a discussion of high order subharmonics, but it has been

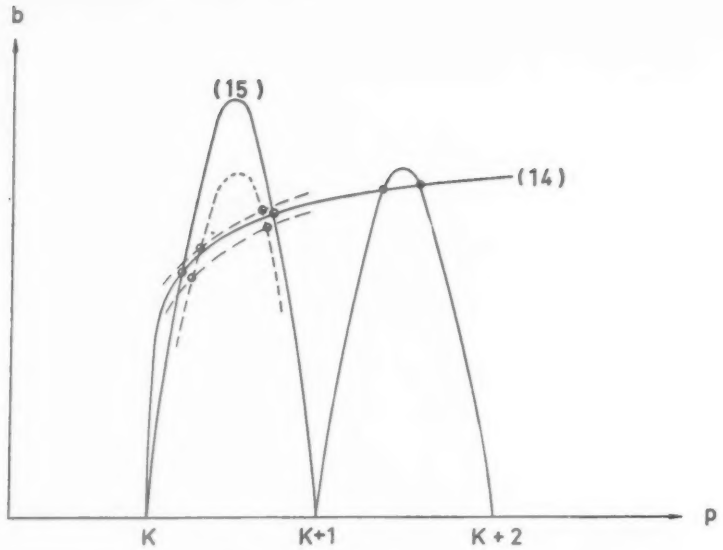

FIG. 2. Construction of response-curves according to (14) and (15). Solid curves: $C = 0$. Dotted curves: $C = C_1$.

used for $k = 2$ with good result. With $k = 1$, $(p - k) \ll 1$, Eq. (13) gives a part of the harmonic response curve, which may be used for determining the function $G(b)$.

Stability. If all points close to a fixed point P_0 move towards P_0 under T, the corresponding solution is stable, if some points move away from P_0 it is unstable. The fixed points can be classified similarly to the singular points of a first order equation[2]. The transformation vector is approximately tangential to the curves defined by $db/d\alpha = A/B$, for which the characteristic equation near a singular point (b_0 , α_0) is

$$\left[\lambda + \frac{\epsilon\kappa}{2}\right]^2 + p^2\left(\frac{\epsilon G(b_0)}{b_0} - \frac{p-k}{p}\right)\left(\epsilon G'(b_0) - \frac{p-k}{p}\right) = 0. \tag{16}$$

Instability occurs when

$$\left(\frac{\epsilon G(b_0)}{b_0} - \frac{p-k}{p}\right)\left(\epsilon G'(b_0) - \frac{p-k}{p}\right) < -\left(\frac{\epsilon\kappa}{2p}\right)^2.$$

At the point where the response curve enters the unstable region $\partial p/\partial b = 0$ and $\partial F/\partial b = 0$. From this point jumps are possible (Fig. 3).

Fig. 4 gives an idea of the general form of subharmonic response curves.

Remarks. Subharmonic oscillations of a given order exist in certain frequency bands.

If several states of subharmonic oscillation are possible at a given frequency the amplitude is largest and the frequency band smallest for the lowest order.

For a given form of nonlinearity the highest order obtainable is a function of the ratio F/κ, which also determines the width of the frequency bands. The condition for

the order k to exist is roughly

$$\epsilon G\left(\frac{2F}{\pi k \kappa}\right) > \frac{F}{\pi k^2 \kappa}.$$

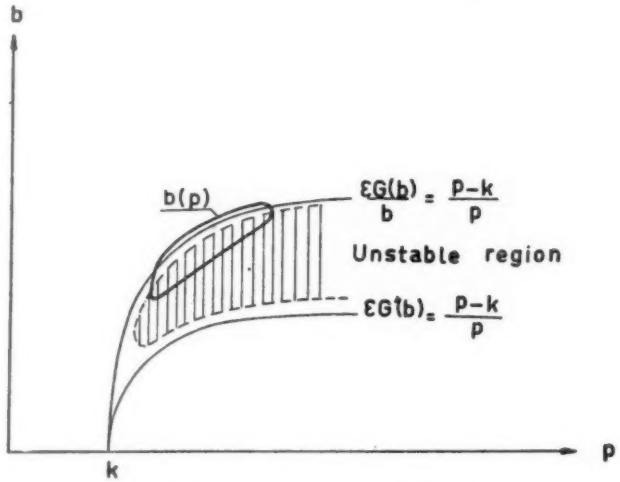

FIG. 3. Stability of subharmonic oscillations.

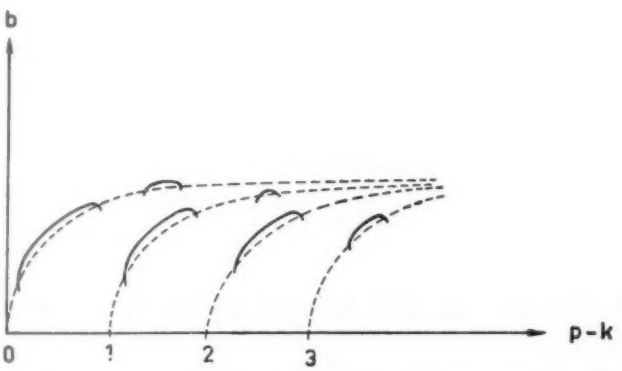

FIG. 4. Amplitude of subharmonic oscillations (solid curves). On branch starting at $p = k$ the frequency of the subharmonic oscillation is p/k.

There is no relation between the analytical form of and the possible values of k, as is sometimes stated[5]. If $f(y)$ is not odd a constant term must be added in Eq. (2).

Experiment. Experiment on subharmonics have recently been made by Messrs. Hansson and Göransson, who have kindly permitted me to quote their results[6]. They

[5]C. Hayashi, Journ. Appl. Phys. **24**, 521, 1953.

[6]K. Göransson and L. Hansson, Tekn. Högskolans Handlingar (in press).

have studied the oscillations in an electric circuit containing iron, where the nonlinearity is essentially of the seventh degree. The damping is reduced by an electronic device, so it has been possible to obtain all subharmonics from the second to the ninth order. For small values of ϵ a good agreement with this theory is obtained. The band character and the existence of an upper limit for k has been clearly demonstrated. The subharmonic state was excited by a voltage pulse of the same magnitude as b.

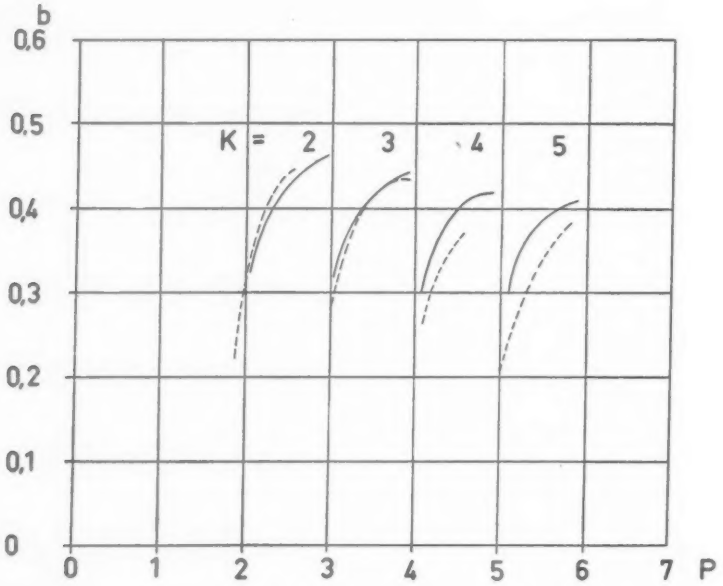

FIG. 5. Experimental curves (dotted) according to K. Göransson and L. Hansson and theoretical values from Eq. (13).

ON THE STABILITY OF THE AXIALLY SYMMETRIC LAMINAR JET*

BY H. G. LEW (*The Pennsylvania State University*)

1. The stability of the axially symmetric laminar jet will be investigated herein subject to rotationally symmetric disturbances. We suppose the laminar jet to be issuing from a small hole with the motion symmetrical about the z axis which is aligned with the jet. The angular position of any point is given by the angle φ measured from the positive x-axis, and r the perpendicular distance from the axis. Thus a set of cylindrical polar coordinates is used.

The steady state flow can be solved in closed form and is given in Goldstein [Ref. 1].

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The distribution of the velocity in the direction of the jet axis is

$$W = \frac{3}{8}\frac{M}{\pi\mu}\frac{1}{z}\frac{1}{(1+\frac{1}{4}\xi^2)^2}, \tag{1}$$

where M is the rate of momentum flowing across a section of the jet and is a constant, and ξ is defined by

$$\xi = \frac{1}{4\nu}\left(\frac{3M}{\pi\rho}\right)^{1/2}\frac{r}{z}.$$

If we select characteristic length and velocity as

$$L = \left(\frac{M}{\mu\nu z^2}\right)^{-1/2} \tag{2}$$

$$W_0 = \frac{3}{8}\frac{M}{\pi\mu}\frac{1}{z}$$

respectively, then the velocity W from (1) becomes:

$$\frac{W}{W_0} = \frac{1}{[1+b(r/L)^2]^2}, \tag{3}$$

where $b = 3/64\pi$.

We consider now the stability of the axially symmetric jet with regard to small rotationally symmetric disturbances of the exponential type. The disturbances are considered small compared with steady-state flow and all non-linear terms of the disturbances are neglected. Moreover, the steady-state flow is to be of the boundary layer type. Thus, we have for the velocities and pressure

$$\begin{aligned} w^* &= W(r) + w(r)\exp[i\alpha(z-ct)], \\ u^* &= \quad\; u(r)\exp[i\alpha(z-ct)], \\ v^* &= \quad\; v(r)\exp[i\alpha(z-ct)], \\ p^* &= P + \quad p(r)\exp[i\alpha(z-ct)], \end{aligned} \tag{4}$$

where W, P are the steady-state quantities, the others are the disturbances, and α(real) the wave number, and $c = c_r + ic_i$ with c_r the wave velocity and c_i the damping or amplification factor. If Eqs. (4) are inserted into the equations of motion [Ref. 1], neglecting non-linear terms of the disturbances, and also assuming that the steady state flow is of the boundary layer type, we obtain the following disturbance equations:

$$ui\alpha(W-c) = -\frac{1}{\rho}p' + \nu\left(u'' + \frac{1}{r}u' - u\alpha^2 - \frac{u}{r^2}\right), \tag{a}$$

$$i\alpha(W-c)v = \nu\left(v'' + \frac{1}{r}v' - v\alpha^2 - \frac{v}{r^2}\right), \tag{b}$$

$$i\alpha(W-c)w + uW' = -\frac{1}{\rho}pi\alpha + \nu\left(w'' + \frac{w'}{r} - w\alpha^2\right), \tag{c}$$

$$(ru)' + wri\alpha = 0, \tag{d}$$

(5)

where the primes denote differentiations with respect to r. The boundary conditions are that the disturbances u, v, w, and p are bounded and that they vanish at infinity. Moreover, all integrals over $(0, \infty)$ are to be convergent.

It is noted that Eq. (5b) is uncoupled from the other equations (5a,c,d). Thus, all discussion of stability with regard to the v-component may be obtained from Equation (5b). The steady-state flow may be shown to be stable to the v-disturbance component in the following way. Following Synge [Ref. 2], we multiply Eq. (5b) by $(r\bar{v})$ (the bar denotes the complex conjugate) and integrate over the interval $(0, \infty)$. Thus we obtain:

$$i\alpha \int_0^\infty (W - c) \cdot v\bar{v}r \, dr = \nu \int_0^\infty \left[v''\bar{v}r + \bar{v}v' - r\bar{v}v\alpha^2 - \frac{v\bar{v}}{r} \right] dr. \tag{6}$$

If we integrate the first term of Eq. (6) by parts, assert the boundary condition $v(+\infty) = 0$, and separate real and imaginary parts, we obtain

$$c_r = \frac{\int_0^\infty Wrv\bar{v} \, dr}{\int_0^\infty rv\bar{v} \cdot dr}$$

and

$$c_i = -\frac{\nu}{\alpha} \int_0^\infty \left[v'\bar{v}' + \bar{v}v\alpha^2 + \frac{v\bar{v}}{r} \right] dr,$$

assuming that v vanishes at least by $r^{-1/2}$ at infinity. Thus c_i is always negative and therefore the jet is stable to the v disturbance. Hence, in this case of rotationally symmetric disturbances, it is sufficient to investigate the case of axially symmetric disturbances.

2. We consider the inviscid case now for the u, w and p components. Setting $\nu = 0$ in Eqs. (5a) and (5c), eliminating the p term by the usual differentiation and subtraction processes, and utilizing the continuity Eq. (5d), we obtain one equation for the u component

$$(W - c)\left[u'' - \alpha^2 u + \left(\frac{u}{r}\right)' \right] - u\left(W'' - \frac{W'}{r} \right) = 0. \tag{7}$$

We may non-dimensionalize Eq. (7) by using the characteristic length and velocity given by Eq. (2); for example, the non-dimensional disturbance velocity and coordinate r are u/W_0, r/L, etc. The resulting equation has the same form as that of Eq. (7), and hereafter, we note that Eq. (7) and all subsequent equations in this section, have been non-dimensionalized and no new equations will be introduced. Here we note the values of the steady state velocity and its derivatives for future use. They are:

$$\begin{aligned} W &= \frac{1}{(1 + br^2)^2}, \\ W'' - \frac{W'}{r} &= \frac{24b^2r^2}{(1 + br^2)^4}. \end{aligned} \tag{8}$$

We can show that for the axially symmetric jet given by Eqs. (8) there exist no self-excited disturbances and no damped disturbances when viscosity is neglected. The proof is as follows:

Consider a transformation for $u(r)$ of the form:

$$u(r) = r^{-1/2}g(r). \tag{9}$$

Insertion of Eq. (9) in the disturbance Eq. (7) leads to

$$g'' + \left(-\frac{3}{4r^2} - \alpha^2\right)g - \frac{W'' - r^{-1}W'(r)}{W - c}\cdot g = 0. \tag{10}$$

Let

$$\begin{aligned} L[g] &= g'' - \left(\frac{3}{4r^2} + \alpha^2\right)g, \\ f(r) &= -\frac{W'' - r^{-1}W'(r)}{W - c}, \end{aligned} \tag{11}$$

then Eq. (10) can be written as:

$$L[g(r)] + \cdot f(r)\cdot g(r) = 0. \tag{12}$$

We assume a non-neutral oscillation so that there is a solution g with c not real. We multiply Eq. (12) by the complex conjugate $\bar{g}$ and integrate over $(0, \infty)$ and subtract its conjugate. Thus

$$\int_0^\infty \{\bar{g}L[g] - gL[\bar{g}]\}\, dr + \int_0^\infty \bar{g}g(f - \bar{f})\, dr = 0 \tag{13}$$

and since

$$\int_0^\infty \{\bar{g}L[g] - gL[\bar{g}]\}\, dr = 0, \tag{14}$$

then

$$\int_0^\infty \bar{g}g(f - \bar{f})\, dr = -2ic_i \int_0^\infty \bar{g}g\, \frac{W'' - r^{-1}W'(r)}{|W - c|^2}\, dr = 0. \tag{15}$$

However, since Eq. (8) shows that $(W'' - W'/r)$ is always positive in $(0, \infty)$, Eq. (15) is satisfied only if $c_i = 0$. Hence there are no self-excited or damped disturbances if viscosity is neglected, and therefore only neutral disturbances need be considered. This result is interesting since any section taken of the jet containing the jet axis leads to a velocity distribution with two inflection points for which non-neutral oscillation may occur if that profile is a two-dimensional one. Of course, here we do not have a true inflection point in that sense since the flow is three dimensional, and rotational symmetric disturbances are considered only.

We shall consider now the case of the neutral mode. Suppose there is a point r_0 such that $[W''(r_0) - W'(r_0)/r_0] = 0$ in the interval $(0, \infty)$, then r_0 is either 0 or ∞ by Eq. (8). At $r_0 = 0$ and ∞ we have, respectively,

$$W(0) = 1 \qquad \text{and} \qquad W(\infty) = 0.$$

Now consider the differential equation

$$g'' + \lambda g + Kg = 0, \tag{16}$$

where

$$K = -\frac{3}{4r^2} - \frac{W'''(r) - r^{-1}W'(r)}{W(r) - W(r_0)}$$

and

$$W(r_0) = 0 \quad \text{or} \quad 1.$$

Equation (16) is equivalent to a variational problem for the first eigenvalue of

$$\lambda = \min \frac{\int_0^\infty (g'^2 - Kg^2)\, dr}{\int_0^\infty g^2\, dr}. \tag{17}$$

We must show that the ratio in Eq. (17) is negative for some function g satisfying the boundary condition. Then $\lambda < 0$ and the minimum will then be below this value. For the case of $W(r_0) = 0$, K will always be negative and therefore all characteristic numbers λ are positive and no neutral oscillation occurs. In the case of $W(r_0) = 1$ at $r_0 = 0$, K may be positive, and there may exist a function such that the ratio in Eq. (17) is negative. Numerical calculations for this case are given in [Ref. 3].

It is interesting to note that the term

$$\frac{1}{r}\left(W'' - \frac{W'}{r}\right) = \frac{d}{dr}\left(\frac{1}{r}\frac{dW}{dr}\right)$$

is the gradient of $-\eta/r = r^{-1}W'(r)$ where η is the vorticity of the basic flow in accordance with the parallel flow assumption. Thus, the condition for neutral oscillation corresponds to the vanishing of this gradient at some point. This is somewhat analogous to the two-dimensional case discussed by Lin in [Ref. 4] where a physical interpretation of this condition for stability is given. However, here the important quantity is η/r and not the vorticity alone.

The extension of these discussions to three-dimensional disturbances should allow for the dependency of the disturbances on the angular variable φ.

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ON SOURCE AND VORTEX OF FLUCTUATING STRENGTH TRAVELLING BENEATH A FREE SURFACE*

By H. S. TAN (*University of Notre Dame*)

In the coordinate system which moves with the travelling source or vortex at constant forward speed c, and under the hypothesis that the resulting fluid motion is irrotational, one can define a disturbance velocity potential $\Phi(x, y, t)$ for the two dimensional fluid motion through the differential equation

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$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0, \tag{1}$$

the boundary condition on free surface $Y(y = 0)$

$$\frac{\partial \Phi}{\partial t} - c\frac{\partial \Phi}{\partial x} = -gY,$$

$$\frac{\partial Y}{\partial t} - c\frac{\partial Y}{\partial x} = \frac{\partial \Phi}{\partial y},$$

i.e.

$$\frac{\partial^2 \Phi}{\partial t^2} - 2c\frac{\partial^2 \Phi}{\partial t\, \partial x} + c^2\frac{\partial^2 \Phi}{\partial x^2} + g\frac{\partial \Phi}{\partial y} = 0, \tag{2}$$

and the physical condition that the upstream disturbance must vanish at infinity, i.e.

$$\lim_{x\to\infty}\left[\left|\frac{\partial \Phi}{\partial x}\right|^2 + \left|\frac{\partial \Phi}{\partial y}\right|^2\right] = 0. \tag{3}$$

When the moving source or vortex executes harmonic fluctuation in strength, by further writing the velocity potential in the form

$$\Phi(x, y, t) = \Re\phi(x, y) \exp(i\omega t), \tag{4}$$

one obtains the following differential system for ϕ:

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0, \tag{1'}$$

$$\nu\frac{\partial \phi}{\partial y} - 2i\tau\nu\frac{\partial \phi}{\partial x} - \nu^2\phi + \tau^2\frac{\partial^2 \phi}{\partial x^2} = 0, \qquad y = 0, \tag{2'}$$

$$\lim_{x\to\infty}\left[\left|\frac{\partial \phi}{\partial x}\right|^2 + \left|\frac{\partial \phi}{\partial y}\right|^2\right] = 0, \tag{3'}$$

where

$$\tau = \omega c/g, \qquad \nu = \omega^2/g.$$

(I) **Pulsating moving source.** Writing the space potential of our moving source (located at $\zeta = \xi + i\eta$) in the form

$$G_1 = \ln|z - \zeta| - \ln|z - \bar{\zeta}| + G(x, y; \xi, \eta), \tag{5}$$

where $G(x, y; \xi, \eta)$ is harmonic in the lower half plane, and noting that at $y = 0$:

$$\frac{\partial}{\partial y}\ln|z - \zeta| = -\frac{\partial}{\partial y}\ln|z - \bar{\zeta}| = \vartheta(z - \bar{\zeta})^{-1},$$

$$\frac{\partial^n}{\partial x^n}\ln|z - \zeta| = \frac{\partial^n}{\partial x^n}\ln|z - \bar{\zeta}|, \tag{6}$$

it is easy to see that G must satisfy the following differential equation at $y = 0$:

$$\nu\frac{\partial G}{\partial y} - 2i\tau\nu\frac{\partial G}{\partial x} - \nu^2 G + \tau^2\frac{\partial^2 G}{\partial x^2} = 2\nu\frac{\partial}{\partial y}\ln|z - \bar{\zeta}|. \tag{7}$$

Now both sides of (7) involve functions harmonic in the lower half plane; (7) holds for entire lower half plane by analytic extension.

From the well-known relation

$$(z - \bar{\zeta})^{-1} = i \int_0^\infty \exp\,[-ik(z - \bar{\zeta})]\, dk, \qquad y + \eta < 0, \tag{8}$$

the following expression can easily be established:

$$2\nu \frac{\partial}{\partial y} \ln |\, z - \bar{\zeta}\,| = -\nu \int_0^\infty \exp\,[k(y + \eta)]\{\exp\,[ik(x - \xi)] + \exp\,[-ik(x - \xi)]\}\, dk. \tag{9}$$

Thus combining (9) and (7), one obtains

$$G = \nu \int_0^\infty \frac{\exp\,[k(y + \eta)]}{\tau^2 k^2 + (2\tau - 1)\nu k + \nu^2} \exp\,[-ik(x - \xi)]\, dk + \nu \int_0^\infty \frac{\exp\,[k(y + \eta)]}{\tau^2 k^2 - (2\tau + 1)\nu k + \nu^2} \exp\,[ik(x - \xi)]\, dk. \tag{10}$$

The disturbance velocity potential of a moving pulsating source can therefore be written in the following form

$$\begin{aligned}\Phi &= \{\ln |\, z - \zeta\,| - \ln |\, z - \bar{\zeta}\,| \\ &\qquad + G + \sum C_n \exp\,[k_n(y + \eta)] \exp\,[\pm ik_n(x - \xi)]\} \exp\,(i\omega t) \\ &= G^*(z, \zeta) \exp\,(i\omega t),\end{aligned} \tag{11}$$

where the k_n's are the poles of the integrands in (10). The constants C_n are to be determined by the physical condition at upstream infinity (3), i.e.

$$\lim_{x \to \infty} \left[\left| \frac{\partial G^*}{\partial x} \right|^2 + \left| \frac{\partial G^*}{\partial y} \right|^2 \right] = 0. \tag{12}$$

(II) Fluctuating moving vortex. Analogous to (I), one can write the space potential of our single moving vortex (located at $\zeta = \xi + i\eta$) in the form

$$H_1 = \theta(z - \zeta) + \theta(z - \bar{\zeta}) + H(x, y; \xi, \eta), \tag{13}$$

where $H(x, y; \xi, \eta)$ is harmonic in the lower half plane. Noting the symmetry of $\theta + \bar{\theta}$, $(\bar{\theta} = \theta(z - \bar{\zeta}))$, at $y = 0$, i.e.

$$\begin{aligned}\frac{\partial}{\partial y}\theta &= \frac{\partial}{\partial y}\bar{\theta} = \Re(z - \bar{\zeta})^{-1}, \\ \frac{\partial^n}{\partial x^n}\theta &= -\frac{\partial^n}{\partial x^n}\bar{\theta},\end{aligned} \tag{14}$$

it can easily be shown that H must satisfy the differential equation

$$\nu \frac{\partial H}{\partial y} - 2i\tau\nu \frac{\partial H}{\partial x} - \nu^2 H + \tau^2 \frac{\partial^2 H}{\partial x^2} = -2\nu \frac{\partial}{\partial y}\bar{\theta}. \tag{15}$$

Now

$$2\nu \frac{\partial \bar{\theta}}{\partial y} = i\nu \int_0^\infty \exp\,[k(y + \eta)][\exp\,[ik(x - \xi)] - \exp\,[-ik(x - \xi)]]\, dk. \tag{16}$$

Thus combining (16) and (15), one obtains

$$H = i\nu \int_0^\infty \frac{\exp\,[k(y+\eta)]}{\tau^2 k^2 - (2\tau+1)\nu k + \nu^2} \exp\,[ik(x-\xi)]\,dk$$

$$- i\nu \int_0^\infty \frac{\exp\,[k(y+\eta)]}{\tau^2 k^2 + (2\tau-1)\nu k + \nu^2} \exp\,[-ik(x-\xi)]\,dk. \qquad (17)$$

The disturbance velocity potential of a moving vortex of fluctuating strength accordingly can be written

$$\Phi_1 = \{\theta + \bar{\theta} + H + \sum D_n \exp\,[k_n(y+\eta)] \exp\,[\pm ik_n(x-\xi)]\} \exp\,(i\omega t)$$

$$= H^*(z, \zeta) \exp\,(i\omega t), \qquad (18)$$

where, as before, the k_n's are the poles of the integrands in (17), and the constants D_n are to be determined by the physical condition at upstream infinity, Eq. (3).

It should be noted, however, that fluctuation in strength of a single travelling vortex does not occur alone, i.e., this fluctuation is necessarily accompanied by a simultaneous shedding of vortices into the wake, of an amount equal and opposite to the increment in travelling vortex strength, so that Helmholtz's law is not violated. Expression (18) accordingly represents only a part of the complete velocity potential of the induced flow field. The complete potential is obtained by simply adding to (18) the contribution due to the distributed wake vortices. Noting that

$$\frac{\partial \Phi_1}{\partial t'} = i\omega H^*(z, \zeta') \exp\,[i\omega(t+t')]$$

and

$$ct' = \xi' - \xi, \qquad c\,dt' = d\xi',$$

where the primed system denotes the running coordinate, it can easily be shown that the latter contribution is given by the following integration

$$\Phi' = -\int_{-\infty}^{0} \frac{\partial \Phi_1}{\partial t'}\,dt' = -\frac{i\omega}{c} \int_{-\infty}^{\xi} H^*(z, \zeta') \exp\left\{i\omega\left[t - \frac{1}{c}(\xi - \xi')\right]\right\} d\xi'. \qquad (19)$$

Hence the complete velocity potential of a single vortex of fluctuating strength travelling beneath a free surface can be written in the following form:

$$\Phi = \left[H^*(z, \zeta) - \frac{i\omega}{c} \int_{-\infty}^{\xi} H^*(z, \zeta') \exp\left[-\frac{i\omega}{c}(\xi - \xi')\right] d\xi'\right] \exp\,(i\omega t). \qquad (20)$$

It might be of interest to remark that a travelling vortex line of fluctuating strength is an idealization of a lifting surface with oscillating angle of attack, thus (20) actually offers a first approximation to the solution of the oscillating hydrofoil. An exact solution to the oscillating hydrofoil requires, of course, the determination of the distribution of fundamental solutions (11) and (20) along the foil contour so that the boundary condition on contour and Kutta condition at the trailing edge be satisfied. In that case it will not be difficult to see that the problem leads directly to a Fredholm type integral equation of the second kind.

AN OPTIMUM IMPLICIT RECURRENCE FORMULA FOR THE HEAT CONDUCTION EQUATION*

By STEPHEN H. CRANDALL (*Massachusetts Institute of Technology*)

The equation $\psi_{xx} = \psi_t$ may be approximately integrated over a finite difference network, $\Delta x = h$, $\Delta t = rh^2$, using

$$\psi_{j,k+1} - r\theta(\psi_{j-1,k+1} - 2\psi_{j,k+1} + \psi_{j+1,k+1}) = \psi_{j,k} + r(1 - \theta)(\psi_{j-1,k} - 2\psi_{j,k} + \psi_{j+1,k}), \qquad (1)$$

where $\psi_{j,k} = \psi(j\Delta x, k\Delta t)$ and $0 \leq \Theta \leq 1$. When $\Theta = 0$, we have the recurrence formula of Schmidt [1], when $\Theta = 1/2$ that of Crank and Nicolson [2] and when $\Theta = 1$ a formula mentioned by O'Brien, Hyman and Kaplan [3]. The truncation error of (1) is ordinarily $0(h^2)$. Milne [4] has shown that when $\Theta = 0$ the truncation error is $0(h^4)$ for $r = 1/6$. Fowler showed that the stability limit for $\Theta = 0$ was in the neighborhood of $r = 1/2$; the exact value depending on the number of spatial subdivisions and the boundary conditions. It is also known [6] that for $\Theta = 0$ some of the finite difference modes have oscillatory time behavior when r is greater than about $1/4$.

It is a straightforward matter to extend these analyses to $0 \leq \Theta \leq 1$. The results are shown in Fig. 1. Each point in the (r, Θ) plane represents a recurrence formula (1). All are implicit except those with $\Theta = 0$. The loci of the stability and oscillation limits

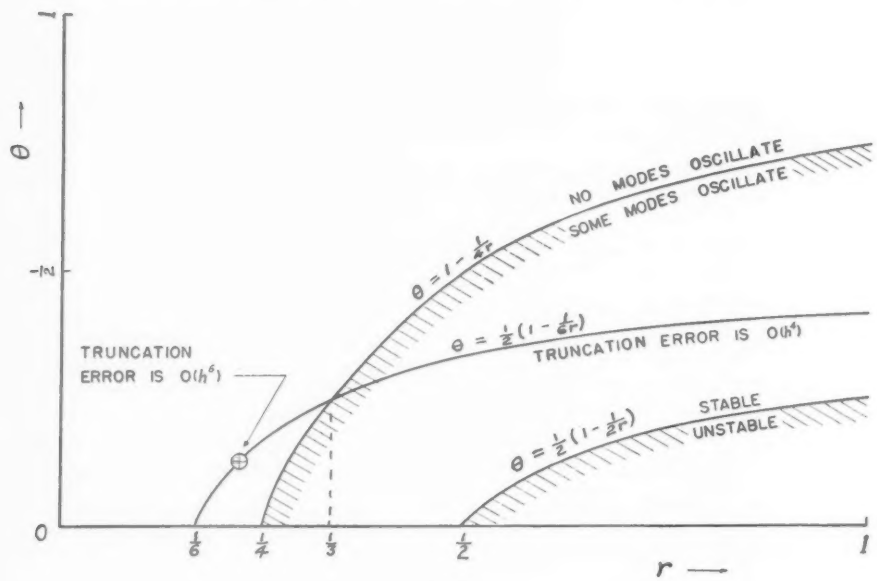

FIG. 1

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are shown along with the family of formulas for which the truncation error is $0(h^4)$. The derivation of these loci is outlined below.

The general solution of (1) in a domain where the boundary conditions at $j = 0$ and $j = M$ are independent of k can be obtained by the method of separation of variables [7] as a sum of products of space modes multiplied by time factors. The space modes are determined by the eigenvalue equation

$$X_{j+1} - (2 - \lambda)X_j + X_{j-1} = 0 \tag{2}$$

and the boundary conditions. The number of modes depends on the number of spatial subdivisions M. The most oscillatory mode has the largest eigenvalue λ_{max}. It is not difficult to show that $\lambda_{max} \to 4$ as $M \to \infty$. Its precise value for finite M depends on M and also on what finite difference approximations [8] have been used at the boundaries.

The time factors T_k which multiply the space modes X_j are

$$T_k = \left(\frac{1 - (1 - \theta)\lambda r}{1 + \theta\lambda r}\right)^k. \tag{3}$$

For (1) to be *stable* in the sense that no solution can become unbounded with increasing k it is necessary for the absolute value of the fraction in (3) to be less than unity for all modes. To obtain the stability limit we set this fraction equal to -1 and insert λ_{max}. Using $\lambda_{max} = 4$ we obtain the locus

$$\theta = \frac{1}{2}\left(1 - \frac{1}{2r}\right) \tag{4}$$

shown in Fig. 1 which is the limiting stability locus as $M \to \infty$.

For all modes to have *non-oscillatory* time behaviour it is necessary for the fraction in (3) to be non-negative. To obtain the oscillation limit we set the numerator of this fraction equal to zero and insert λ_{max}. Again using $\lambda_{max} = 4$ we obtain the locus

$$\theta = 1 - \frac{1}{4r} \tag{5}$$

shown in Fig. 1.

To investigate the truncation error of (1) the values of an exact solution to $\psi_{xx} = \psi_t$ at neighboring net points are represented by means of Taylor's series centered at $(j\Delta x, k\Delta t)$. When these are substituted in (1) the remainder terms of the Taylor's series are

$$h^2\left(\frac{r}{2} - \frac{1}{12} - r\theta\right)\psi_{tt} + h^4\left(\frac{r^2}{6} - \frac{1}{360} - \frac{r^2\theta}{2} - \frac{r\theta}{12}\right)\psi_{ttt} + 0(h^6), \tag{6}$$

where we have used the identities

$$\psi_{tt} = \psi_{txx} = \psi_{xxxx} \qquad \psi_{ttt} = \psi_{ttxx} = \psi_{txxxx} = \psi_{xxxxxx}. \tag{7}$$

Thus the truncation error is ordinarily $0(h^2)$ but is $0(h^4)$ when

$$\theta = \frac{1}{2}\left(1 - \frac{1}{6r}\right). \tag{8}$$

Among the sets of values which satisfy (8) there is one set which also makes the coefficient of h^4 in (6) vanish. It is easily verified that this set is

$$r = \frac{(5)^{1/2}}{10} = 0.2236, \qquad \theta = \frac{3 - (5)^{1/2}}{6} = 0.1273. \tag{9}$$

The truncation error of (1) with the particular values (9) is therefore $0(h^6)$. This should provide a superior recurrence formula for integrating $\psi_{xx} = \psi_t$. As usual with implicit formulas it will be necessary to invert but this can be done once and for all as soon as the number of spatial subdivisions has been chosen.

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ON A METHOD OF GENERATING THREE-DIMENSIONAL POTENTIAL FLOWS FROM TWO-DIMENSIONAL ONES*

By CHIA-SHUN YIH (*Iowa Institute of Hydraulic Research*)

In 1951, A. Wintner[1] presented a method of deriving three-dimensional potential flows from two-dimensional ones. This method was found by Weingarten[2] over sixty years ago and rediscovered by Levi-Civita[3] a decade later. Whereas it is surprising that the flows thus derived should have been forgotten in the intervening years, it is indeed strange that another method for the same purpose, which is very straightforward and simple, never seems to have been mentioned. In the following, this new method, which differs essentially from the geometrical method of Weingarten, will be presented together with some examples illustrating its practical applications.

For irrotational flow, the curl of the velocity vector is zero, and the velocity com-

*Received Dec. 2, 1954.

[1]A. Wintner, *On a geometrical method of deriving three-dimensional harmonic flows from two-dimensional ones*, Quart. Appl. Math. 9, 102-105 (1951).

[2,3]See Ref. 1.

ponents u, v, and w in the directions of x, y, and z can be derived from a potential ϕ in the following way:

$$u = -\phi_x , \qquad v = -\phi_y , \qquad w = -\phi_z , \tag{1}$$

in which the subscripts denote partial differentiation. From Eqs. (1) and the equation of continuity

$$u_x + v_y + w_z = 0 \tag{2}$$

it follows that the velocity potential satisfies the Laplace equation

$$\phi_{xx} + \phi_{yy} + \phi_{zz} = 0. \tag{3}$$

If one puts

$$\phi = f(x, y) + g(y, z) + h(z, x) \tag{4}$$

and demands that

$$f_{xx} + f_{yy} = 0, \qquad g_{yy} + g_{zz} = 0, \qquad h_{zz} + h_{xx} = 0, \tag{5}$$

Eq. (3) is evidently satisfied. Hence the

Theorem: Harmonic functions of any two of the variables x, y, and z can be superposed to form a harmonic function of all three variables; or, physically, two-dimensional potential flows parallel to mutually perpendicular planes can be superposed to form a three-dimensional potential flow.

As is well known, harmonic functions of two variables can be obtained from an analytic function of the corresponding complex variable. For instance, the real parts as well as the coefficients of the imaginary number in the analytic functions $F(\alpha)$, $G(\beta)$, and $H(\gamma)$ in which

$$\alpha = x + iy, \qquad \beta = y + iz, \qquad \gamma = z + ix$$

are all harmonic functions, and any number of them can be used for superposition to form a velocity potential for three-dimensional flow. However, to conform with established usage, the real parts of $F(\alpha)$, $G(\beta)$, and $H(\gamma)$ will be taken to be the functions f, g, and h in Eq. (4).

A few examples will now be given. If

$$F(\alpha) = \alpha + \frac{a^2}{\alpha} , \qquad G(\beta) = 0, \qquad H(\gamma) = \frac{ia^2}{\gamma} ,$$

superposition of the two-dimensional flows gives the flow across a solid body consisting of a hub and four spokes. The cross sections of these spokes are approximately circles of radius a at large distances from the hub. The velocity at large distances from the solid body is uniform and in the x-direction. Of course, the terms a^2/α and ia^2/γ can be replaced by suitable functions so that the cross section of the vertical spokes and that of the horizontal ones may have any desirable shapes at large distances from the hub, so long as the corresponding two-dimensional flows are known.

The flow obtained from

$$F(\alpha) = \alpha + C \coth \frac{\pi\alpha}{b} , \qquad G(\beta) = 0, \qquad H(\gamma) = -C' \coth \frac{\pi\gamma i}{d} ,$$

in which C and C' are real and positive, is the flow across a rectangular grid. The cross sections of the bars far from the intersections are roughly circular, particularly if b is large compared with C, and d is large compared with C'.

The flow corresponding to

$$F(\alpha) = C\alpha^2, \qquad G(\beta) = D\beta^2, \qquad H(\gamma) = E\gamma^2,$$

is one for which the coordinate planes are not pierced by streamlines and are therefore "streamline surfaces" perpendicular to equi-potential surfaces.

Three-dimensional flows constructed by the present method belong to a special class, of which two-dimensional flows form a sub-class. Axisymmetric flows, however, do not belong to this class.

The reviewer of the original manuscript of this paper suggested that the expressions of the velocity components in terms of the complex conjugates of f, g, and h be also given. If the latter are denoted by f', g', and h', then the Cauchy-Riemann equations are

$$f_x = f'_y\,, \qquad f_y = -f'_x\,; \qquad g_y = g'_x\,, \qquad g_x = -g'_y\,; \qquad h_x = h'_x\,, \qquad h_x = -h'_x \tag{6}$$

and one has, from Eqs. (1), (4), and (6):

$$u = h'_x - f'_y\,, \qquad v = f'_x - g'_x\,, \qquad w = g'_y - h'_x\,. \tag{7}$$

In vector form, Eq. (6) can be written as

$$\mathbf{V} = -\operatorname{curl} \mathbf{A}$$

in which

$$\mathbf{A} = \mathbf{i}g' + \mathbf{j}h' + \mathbf{k}f'$$

($\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ being unit vectors along the coordinate axes) is the vector potential for the flow for which ϕ is the scalar potential.

It may be remarked without further discussion that two-dimensional potential flows which are not parallel to mutually perpendicular planes can also be superimposed to form a three-dimensional potential flow.

A NOTE ON THE MEAN VALUE OF RANDOM DETERMINANTS*

By RICHARD BELLMAN (*The Rand Corporation*)

1. Introduction. In a recent paper, [1], Nyquist, Rice and Riordan discussed the problem of determining the expected values of powers of a random determinant. Here a random determinant, D_n, is defined to be

$$D_n = |\, x_{ij} \,|, \qquad i, j, = 1, 2, \cdots, n,$$

where the x_{ij} are independent random variables.

The purpose of the present note is to give an explicit representation for $E(D_n^k)$ in terms of the characteristic functions of the x_{ij}. These need not be identical.

At the moment we are merely interested in presenting an expression which will

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yield a systematic technique for obtaining the moments numerically. In a subsequent paper devoted to various theoretical aspects such as asymptotic behavior we shall discuss the problem in greater detail. For the case of identical distributions, the problem is closely connected with the study of invariants of the symmetric group. The operator we employ below is related to the operator of Capelli discussed in Weyl's book on the classical groups.

2. A useful operator. Let us consider the operator Θ_n defined as

$$\Theta_n = | \partial/\partial z_{kl} |, \qquad k, 1 = 1, 2, \cdots, n, \tag{2.1}$$

where the z_{kl} are independent variables. Thus

$$\Theta_1 = \partial/\partial z_{11},$$

$$\Theta_2 = \frac{\partial}{\partial z_{11}}\frac{\partial}{\partial z_{22}} - \frac{\partial}{\partial z_{12}}\frac{\partial}{\partial z_{21}}, \tag{2.2}$$

and so on.

Let X represent the matrix (x_{kl}) and Z the matrix (z_{kl}). Then we have

$$\exp [itr(XZ^T)] = \exp (i \sum_{k,l} x_{kl}z_{kl}) \tag{2.3}$$

(Here Z^T is the transpose of Z, and tr (Z) is the trace of Z).
The basic identity we shall employ below is

$$\Theta^k[\exp [itr(XZ^T)]] = i^{nk}D_n^k \exp [itr(XZ^T)], \tag{2.4}$$

for $k = 1, 2, \cdots$.**

3. $E(D_n^r)$. Taking the expected value of both sides in (2.4), we obtain the result

$$\Theta_n^r\left[\prod_{k,l=1}^{n} \phi_{kl}(z_{kl})\right] = i^{nr}E(D_n^k \exp [itr(XZ^T)]), \tag{3.1}$$

where

$$\phi_{kl}(z) = \int_{-\infty}^{\infty} e^{izx}\, dG_{kl}(x), \tag{3.2}$$

is the characteristic function of the random variable x_{kl}. Setting $z_{kl} = 0$, we obtain the result

$$i^{nr}E(D_n^r) = \Theta_n^r\left[\prod_{k,l=1}^{n} \phi_{kl}(z_{kl})\right]_{z_{kl}=0}. \tag{3.3}$$

4. Identical distributions. If the variables are identically distributed and symmetric about zero, we may write

$$\phi(z_{kl}) = \exp (a_1z_{kl}^2 + a_2z_{kl}^4 + \cdots), \tag{4.1}$$

obtaining as a consequence in place of (3.3) the result

$$i^{nk}E(D_n^k) = \Theta_n^k[\exp (a_1 \sum_{k,l} z_{kl}^2 + a_2 \sum_{k,l} z_{kl}^4 + \cdots)]_{z_{kl}=0} \tag{4.2}$$

**This is a well-known device in the theory of matric automorphic functions.

From this representation the value of $E(D_n^k)$ may be obtained by retaining in the above expression only the terms that yield a non-zero value after z_{kl} has been set equal to zero.

A particularly interesting case is that where $x_{kl} = \pm 1$ with equal probability. Then

$$i^{nk}E(D_n^k) = \Theta_n^k\left[\prod_{k,l=1}^{n}(\cos z_{kl})\right]_{z_{kl}=0}. \tag{4.3}$$

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AN INEQUALITY FOR THE FIRST EIGENVALUE OF AN ORDINARY BOUNDARY VALUE PROBLEM*

By PHILIP HARTMAN and AUREL WINTNER (*The Johns Hopkins University*)

Let both coefficient functions of the differential equation

$$y'' + g(x)y' + f(x)y = 0 \tag{1}$$

be real-valued and continuous on the interval $a \leq x \leq b$ and, unless $g'(x)$ is not involved (as it is not in (4) below), suppose that the coefficient of y' has a continuous first derivative $g'(x)$. Consider the boundary condition

$$y(a) = 0, \qquad y(b) = 0. \tag{2}$$

A solution $y(x)$ of (1) satisfying (2) is the trivial solution,

$$y(x) \equiv 0. \tag{3}$$

In what follows, conditions on the function pair (g, f) will be considered which assure that (1) has no solution $y(x)$, distinct from (3), satisfying (2).

Such a condition is known to be

$$f \leq 0 \tag{4}$$

(with an arbitrary g). Another such condition is

$$f - \tfrac{1}{2}g' \leq 0. \tag{5}$$

Still another one is

$$f - g' \leq 0. \tag{6}$$

(It is understood that each of these three conditions is required for all values of x on the interval $a \leq x \leq b$.) Actually, the sufficiency of (4), (5) and (6) is contained in the results of Paraf, Picard and Lichtenstein, respectively, on (elliptic) *partial* differential equations [1]. The method proving the sufficiency of Lichtenstein's condition (6) is quite different from that proving the sufficiency of Picard's condition (5) or of the more primitive condition (4), and it is clear that no one of the three conditions (4)-(6) need be satisfied if the other two are satisfied.

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A comparison of (5) with (6) on the one hand and with (4) on the other will suggest that the $\frac{1}{2}$ in (5) can be replaced by any constant between $\frac{1}{2}$ and 1 or between 0 and $\frac{1}{2}$. But the methods referred to do not reveal the posibility of such generalization. It turns out, however, that this generalization is correct; in other words, that it is sufficient to assume, for all x on the interval $a \leq x \leq b$ and for some $\theta = \text{const.}$, that

$$f(x) - \theta g'(x) \leq 0, \qquad \text{where} \qquad 0 \leq \theta \leq 1. \tag{7}$$

According to Picard, his criterion (5) can be generalized to the following requirement: For some real constant c, the inequality

$$(cg - \tfrac{1}{2}g)^2 + (f - cg') \leq 0 \tag{8}$$

is satisfied at every point of the interval $a \leq x \leq b$ (in fact, (8) results if the function which Picard [2] denotes by $\lambda = \lambda(x)$ is chosen to be the function $cg(x)$, where $c = \text{const.}$ is arbitrary). If (8) is written in the form

$$f(x) - cg'(x) + (c - \tfrac{1}{2})^2 g^2(x) \leq 0, \tag{9}$$

it is seen that the sufficiency of (7) cannot be concluded from it. The converse inference is also impossible (unless $0 \leq c \leq 1$).

It will, however, be shown that condition (9) can be replaced by the following requirement: For some real constant c,

$$f(x) - cg'(x) + c(c - 1)g^2(x) \leq 0 \tag{10}$$

holds at every point of the interval $a \leq x \leq b$. In other words, it will be shown that the existence of a real constant c satisfying (10) is sufficient in order that (3) be the only solution of (1) satisfying (2).

This will imply the sufficiency of (7). In fact, if $0 \leq c \leq 1$, then $c(c - 1) \leq 0$. Hence, if $c = \theta$, where $0 \leq \theta \leq 1$, then (7) is sufficient for (10).

The proof for the sufficiency of (10) proceeds as follows. Choose any real constant c and, in terms of any solution $y = y(x)$ of (1), define a function $Y = Y(x)$ by placing

$$Y(x) = y(x) \exp\left[c \int_a^x g(t)\, dt\right]. \tag{11}$$

Then (2) is equivalent to

$$Y(a) = 0, \qquad Y(b) = 0, \tag{12}$$

and (3) to

$$Y(x) \equiv 0, \tag{13}$$

finally (1) to a differential equation of the same form as (1), say

$$Y'' + G(x)Y' + F(x)Y = 0. \tag{14}$$

In fact, if (11) is written in the form $y = Y \exp(- \cdots)$ and if this y is substituted into (1), an easy calculation shows that (1) appears in the form (14), with

$$G = (1 - 2c)g \tag{15}$$

and

$$F = f - cg' + c(c - 1)g^2. \tag{16}$$

Identify (14) with (1) and apply the criterion (4). It then follows that (13) is the only solution of (14) satisfying (12) if the condition $F \leq 0$, which corresponds to (4), is satisfied. But the definition (16) shows that the condition $F \leq 0$ is identical with assumption (10). Since (12) is equivalent to (2), and (13) to (3), the proof is complete.

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A NOTE ON THE TRANSVERSE VIBRATION OF A TUBE CONTAINING FLOWING FLUID*

By GEORGE H. HANDELMAN† (*Carnegie Institute of Technology*)

1. Introduction. The problem of determining the lowest frequency of vibration of a tube containing flowing fluid has recently received considerable attention. Long [1][1] has considered the tube as a beam and has calculated the frequencies for various end conditions by a power series method. He has pointed out that care must be taken in evaluating the resulting high order determinants to avoid erroneous results. Niordson [2] has given a very elegant treatment of the problem based on shell theory and has derived the beam equation as one approximation. Furthermore, for the case of simply supported ends, he has given closed analytic solutions for two limiting cases of the parameters and numerical results for other values. It is the purpose of this note to show that it is possible to determine the nature of the frequencies for the various end conditions used by Long for two ranges of the flow velocity solely from the structure of the differential equation without determining specific solutions.

Long has written the differential equation governing the deflection of the beam in the following non-dimensional form:

$$F^{IV} + ku^2F^{II} + iuDF' - D^2F = 0, \tag{1}$$

where F is the deflection and u the flow velocity (both in non-dimensional terms). Roman numerals denote differentiation with respect to x ($0 \leqq x \leqq 1$) and D is the eigenvalue found by removing the time dependent term $\exp(iDt)$ from the original partial differential equation. The boundary conditions considered are

Both ends simply supported: $F(0) = F''(0) = F(1) = F''(1) = 0,$

Both ends fixed: $F(0) = F'(0) = F(1) = F'(1) = 0,$

Fixed-simple ends: $F(0) = F'(0) = F(1) = F''(1) = 0,$

Fixed-free ends: $F(0) = F'(0) = F''(1) = F'''(1) = 0.$

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[1]Numbers in square brackets refer to the list of references given at the end of this note.

2. Small flow velocities. The behavior of the frequencies for small velocities of flow can be determined by the usual perturbation technique [3]. It should be observed that the differential equation (1) and boundary conditions specified do not form a self-adjoint problem; however, such will be the case for $u = 0$.

Consider expansions of the form

$$F = F_0(x) + uF_1(x) + u^2F_2(x) + \cdots$$

$$D = D_0 + uD_1 + u^2D_2 + \cdots .$$

If these are inserted in Eq. (1) and the coefficients of each power of u equated to zero, the following equations for the first three terms result:

$$F_0^{IV} - D_0^2F_0 = 0 \qquad (u^0 \text{ term}), \qquad (2)$$

$$F_1^{IV} - D_0^2F_1 = 2D_0D_1F_0 - iD_0F_0' \qquad (u^1 \text{ term}), \qquad (3)$$

$$F_2^{IV} - D_0^2F_2 = -kF_0'' + 2D_0D_1F_1 + 2D_0D_2F_0 - iD_1F_0' - iD_0F_1' \qquad (u^2 \text{ term}). \qquad (4)$$

Both sides of Eq. (3) are now multiplied by F_0 and the resulting expression is integrated from 0 to 1. If this equation is now integrated by parts and account taken of the boundary conditions, it is found that for all cases

$$\int_0^1 F_0(F_1^{IV} - D_0^2F_1)\,dx = \int_0^1 F_1(F_0^{IV} - D_0^2F_0)\,dx = 2D_1D_0\int_0^1 F_0^2\,dx - iD_0\frac{F_0^2}{2}\Big|_0^1.$$

The left-hand side vanishes by virtue of Eq. (2). Furthermore, the first three sets of boundary conditions require that $F(0) = F(1) = 0$. Hence, in these cases $D_1 = 0$. This result agrees with that found by Niordson in the simply supported case, for his computations show that if D is plotted as a function of u, the curve has a horizontal tangent at $u = 0$.

On the other hand, for the fixed-free end, $F_0(1) \neq 0$. Hence,

$$D_1 = \frac{iF_0^2(1)}{4\int_0^1 F_0^2\,dx} = ip,$$

where $p > 0$. Thus the time dependent term becomes

$$\exp\,[i(D_0 + uD_1)t] = \exp\,[i(D_0 + iup)t] = \exp\,(iD_0t)\,\exp\,(-upt).$$

Hence damping takes place when terms through the first order are retained. This agrees with the calculations made by Long using polynomial expansions.

If one of the first three sets of boundary conditions holds (denoted as "supported end cases" in the following), the coefficient D_2 can be obtained by multiplying Eq. (4) by F_0, integrating from 0 to 1, and noting that $D_1 = 0$. Again, after integration by parts, this equation becomes

$$0 = k\int_0^1 F_0'^2\,dx + 2D_0D_2\int_0^1 F_0^2\,dx - iD_0\int_0^1 F_0F_1'\,dx. \qquad (5)$$

The term $\int_0^1 F_0F_1'\,dx$ can be examined by noting that if F_1 is written as the complex function $F_1 = y + iv$, the real and imaginary parts satisfy

$$y^{IV} - D_0^2y = 0, \qquad (6)$$

$$v^{IV} - D_0^2v = -D_0F_0',$$

together with one of the sets of boundary conditions for supported ends given in Sec. 1. Comparison with Eq. (2) then shows that $y = F_0$. Thus

$$\int_0^1 F_0 F_1' \, dx = \int_0^1 F_0 F_0' \, dx + i \int_0^1 F_0 v' \, dx$$

$$= \frac{F_0^2}{2} \Big|_0^1 + iF_0 v \Big|_0^1 - i \int_0^1 F_0' v \, dx = -i \int_0^1 F_0' v \, dx.$$

The sign of the last term can be determined by multiplying Eq. (6) by v and integrating from 0 to 1. Subsequent integration by parts yields

$$-D_0 \int_0^1 F_0' v \, dx = \int_0^1 v(v^{IV} - D_0^2 v) \, dx = \int_0^1 (v''^2 - D_0^2 v^2) \, dx. \tag{7}$$

Now the minimum principle corresponding to Eq. (2) states that for all functions w satisfying suitable differentiability requirements and the boundary conditions, $D_0^2 = \min \int_0^1 w''^2 \, dx / \int_0^1 w^2 \, dx$. The function v is such an admissible function and hence the right-hand side of (7) is always greater than or equal to zero. Therefore, $-D_0 \int_0^1 F_0' v \, dx \geq 0$. This remark and Eq. (5) then show that

$$D_2 = -\frac{k \int_0^1 F_0'^2 \, dx + D_0 \int_0^1 F_0' v \, dx}{2D_0 \int_0^1 F_0^2 \, dx} \leqq 0.$$

Thus for terms up to and including u^2, the frequency for the supported end cases decreases as u increases and no damping occurs. This is in agreement with the numerical results previously obtained.

3. Velocities near the critical velocity. Niordson has observed in the simply supported case, that there is a critical velocity u for which the frequency D vanishes. The equation for this critical velocity is obtained by setting $D = 0$ in Eq. (1). The resulting equation is just the ordinary column buckling equation with ku^2 playing the role of the buckling load. Niordson noted that the bar will be unstable for flow speeds greater than this critical velocity.

In similar fashion, it is seen that there will be a critical velocity for the other supported end cases which can be found from the corresponding buckling problems. On the other hand, there is no critical velocity for the fixed-free case since the differential equation $F^{IV} + ku^2 F'' = 0$ has only trivial solutions for these boundary conditions [4]. For this case then, there is no special velocity u in the neighborhood of which a radical change in the analytic character of D can occur.

The behavior of the frequency D for velocities near the critical velocity in the supported end cases can be readily determined by a perturbation solution in terms of powers of D. If

$$u = u_0 + u_1 D + u_2 D^2 + \cdots,$$

$$F = F_0 + F_1 D + F_2 D^2 + \cdots,$$

the equations for the first three terms obtained from Eq. (1) are

$$F_0^{IV} + ku_0^2 F_0'' = 0 \qquad (D^0 \text{ term}), \tag{8}$$

$$F_1^{IV} + ku_0^2 F_1'' = -2ku_0 u_1 F_0'' - iu_0 F_0' \qquad (D^1 \text{ term}), \tag{9}$$

$$F_2^{IV} + ku_0^2 F_2'' = -2ku_0 u_1 F_1'' - ku_1^2 F_0'' - 2ku_0 u_2 F_0'' - iu_1 F_0' - iu_0 F_1' + F_0 \qquad (D^2 \text{ term}). \tag{10}$$

The coefficient u_1 can be found by multiplying Eq. (9) by F_0 and integrating from 0 to 1. Integration by parts yields

$$0 = -2ku_0u_1F_0F_0'\Big|_0^1 + 2ku_0u_1\int_0^1 F_0'^2\,dx - iu_0\frac{F_0^2}{2}\Big|_0^1.$$

The first and third terms vanish by virtue of the boundary conditions and therefore $u_1 = 0$. This result states that if u is considered as a function of D in the neighborhood of the critical velocity, the curve will have a horizontal tangent. Alternatively, if D is considered as a function of u (as in Sec. 2) in the same neighborhood, the curve will have a vertical tangent. This agrees with Niordson's results for the simply supported case. Furthermore, it indicates why an expansion in powers of D is appropriate rather than one in powers of perturbations from the critical velocity.

The coefficient u_2 is determined by multiplying Eq. (10) by F_0, noting that $u_1 = 0$, and integrating from 0 to 1. Partial integration leads to

$$0 = 2ku_0u_2\int_0^1 F_0'^2\,dx + \int_0^1 F_0^2\,dx + iu_0\int_0^1 F_1F_0'\,dx,$$

and the character of u_2 is then determined by examining the properties of $iu_0\int_0^1 F_1F_0'\,dx$. Let F_1 be written as $F_1 = \xi + i\eta$. If Eq. (9) is then split into its real and imaginary parts

$$\xi^{IV} + ku_0^2\xi'' = 0, \tag{11}$$

$$\eta^{IV} + ku_0^2\eta'' = -u_0F_0'. \tag{12}$$

Comparison of Eq. (11) with Eq. (8) shows that $\xi = F_0$; therefore,

$$iu_0\int_0^1 F_1F_0'\,dx = iu_0\frac{F_0^2}{2}\Big|_0^1 - u_0\int_0^1 \eta F_0'\,dx = -u_0\int_0^1 \eta F_0'\,dx.$$

Equation (12) can then be multiplied by η and integrated from 0 to 1 to yield

$$-u_0\int_0^1 \eta F_0'\,dx = \int_0^1 \eta(\eta^{IV} + ku_0^2\eta'')\,dx = \int_0^1 (\eta''^2 - ku_0^2\eta'^2)\,dx.$$

The minimum principle associated with Eq. (8) states that for all functions w satisfying suitable differentiability conditions and the boundary conditions, $ku_0^2 = \min \int_0^1 w''^2\,dx / \int_0^1 w'^2\,dx$. The function η is such an admissible function. Hence

$$p = -u_0\int_0^1 \eta F_0'\,dx \geqq 0,$$

and

$$u_2 = -\frac{\int_0^1 F_0^2\,dx - p}{2ku_0\int_0^1 F_0'^2\,dx} < 0.$$

If terms through D^2 only are retained, the behavior in the neighborhood of the critical velocity u_0 is expressed by

$$u - u_0 = u_2D^2 = -|\,u_2\,|\,D^2.$$

If the flow velocity is less than the critical velocity, $u < u_0$, $D = \pm\,(|\,u - u_0\,|/|\,u_2\,|)^{1/2}$. Thus D is real and there is neither damping nor exponential growth. On the other hand, if the flow velocity is greater than the critical velocity, $u > u_0$, $D = \pm\, i(|\,u - u_0\,|/|\,u_2\,|)^{1/2}$. In this instance, D is imaginary and the time dependent factor consists of an

exponentially growing term as well as a damping term. With the exception of very special initial conditions, both terms will be present and the tube will be unstable.

4. Further remarks. It appears that the behavior of the higher order perturbation terms cannot be obtained as simply as those discussed above. These may require explicit determination of the function F_0 , F_1 , F_2 , $\cdots$. It should be noted, however, that if F_0 , $\cdots$, F_{n-1} have been found, D_n can be determined by quadratures. Furthermore, the differential equation for F_n will be of the form

$$F_n^{IV} - D_0^2 F_n = \text{previously determined functions.}$$

The Green's function for this equation, in the case of simply supported ends is known [5], and can be determined for the other boundary conditions by standard methods. Thus F_n can be found by integration.

These comments indicate that the perturbation terms can be computed step-by-step by quadratures. Furthermore, in the supported end cases the critical velocity can be determined beforehand and perturbation from this point in powers of D will serve as a check on the perturbation solution in terms of powers of u.

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[3] Lothar Collatz, *Eigenwertprobleme*, Chelsea Publishing Co., New York, 1948, p. 305 ff.

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ON AN OSCILLATION CRITERION OF DE LA VALLÉE POUSSIN*

By PHILIP HARTMAN AND AUREL WINTNER (*The Johns Hopkins University*)

An oscillation criterion of de la Vallée Poussin[1] on homogeneous, linear differential equations of order n, when particularized[2] to $n = 2$, runs as follows: Let both coefficient functions of

$$x'' + g(t)x' + f(t)x = 0 \tag{1}$$

be real-valued and continuous on a t-interval and suppose that (1) has a solution $x(t) \not\equiv 0$ which vanishes for at least two points of that t-interval, say at $t = 0$ and at $t = h > 0$ (so that

$$x(0) = 0, \qquad x(h) = 0, \tag{2}$$

where, without loss of generality, $x(t) \neq 0$ when $0 < t < h$). Then

$$1 < M_1 h + M_2 h^2/2, \tag{3}$$

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[1] G. Sansone, *Equazioni differenziali nel campo reale*, vol. 1, 1948, p. 183.

[2] F. Tricomi, *Equazioni differenziali*, 1948, p. 110.

where

$$M_1 = \max_{0 \leq t \leq h} | g(t) |, \qquad M_2 = \max_{0 \leq t \leq h} | f(t) |. \tag{4}$$

It will be shown in this note that (3) can be improved to

$$1 < M_1 h/2 + M_2 h^2/6. \tag{5}$$

Incidentally, it will also follow that the assumption that $g(t)$ and $f(t)$ are real-valued, an assumption used in de la Vallée Poussin's proof, can be omitted.

It will also follow that, instead of assuming (5) for the numbers (4), it is sufficient to assume

$$1 < \int_0^h | g(t) | \, dt + \frac{h}{4} \int_0^h | f(t) | \, dt \tag{6}$$

for the coefficient functions g, f of (1).

In order that (1) has a solution $x(t) \not\equiv 0$ satisfying (2), a necessary condition for (1), containing both (5) and (6), can be formulated as follows:

$$h < \max \left\{ \int_0^h t \, | g(t) | \, dt, \qquad \int_0^h (h - t) \, | g(t) | \, dt \right\} + \int_0^h t(h - t) \, | f(t) | \, dt. \tag{7}$$

If $g(t) \equiv 0$, then (1) reduces to

$$x'' + f(t)x = 0 \tag{8}$$

and (6) to

$$4 < h \int_0^h | f(t) | \, dt. \tag{9}$$

Hence the criterion (7) to be proved generalizes the following well-known fact, which goes back to Liapounoff[3]: If $f(t)$ is continuous for $0 \leqq t \leqq h$, then a necessary condition for (8) to have a solution ($\not\equiv 0$) satisfying (2) is (9).

The proof of the statement concerning (7) depends on a device used by Nehari[4] for a similar purpose and proceeds as follows:

First, if $x(t)$, where $0 \leqq t \leqq h$, is any function possessing a continuous second derivative and satisfying (2), then it is readily verified that

$$hx(t) = (h - t) \int_0^t sx''(s) \, ds + t \int_t^h (h - s)x''(s) \, ds$$

is an identity for $0 \leqq t \leqq h$. A differentiation gives

$$hx'(t) = -\int_0^t sx''(s) \, ds + \int_t^h (h - s)x''(s) \, ds. \tag{10}$$

Next, it is clear from (2) that both μt and $\mu(h - t)$ are majorants for $| x(t) |$ if $0 \leqq t \leqq h$ and

$$\mu = \max_{0 \leq t \leq h} | x'(t) |. \tag{11}$$

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[3]P. Hartman and A. Wintner, *On an oscillation criterion of Liapounoff*, Amer. J. Math. **73**, 885–890 (1951), where further references will be found.

[4]Z. Nehari, *On the zeros of solutions of second order linear differential equations*, Amer. J. Math. **76**, 690 (1954).

Thus

$$| x(t) | \leqq \mu\varphi(t), \tag{12}$$

where

$$\varphi(t) = \min (t, h - t), \tag{13}$$

and it is clear that the $\leqq$ in (12) is a $<$ for some t.

If x'' is substituted from (1) into (10) (as $-gx' - fx$), it is seen from (11) and (12) that

$$h \mid x'(t) \mid \leqq \mu \int_0^t s(\mid g(s) \mid + \varphi(s) \mid f(s) \mid) \, ds + \mu \int_t^h (h - s)(\mid g(s) \mid + \varphi(s) \mid f(s) \mid) \, ds. \tag{14}$$

Let the maximum of $| x'(t)|$ on $0 \leqq t \leqq h$ be attained at $t = t_0$. Then (14) and the remark made after (13) imply that

$$h < \int_0^{t_0} s(\mid g(s) \mid + \varphi(s) \mid f(s) \mid) \, ds + \int_{t_0}^h (h - s)(\mid g(s) \mid + \varphi(s) \mid f(s) \mid) \, ds, \tag{15}$$

since $| x'(t_0)| = \mu$, by (11), and since $\mu \neq 0$, by (2), where $x(t) \not\equiv 0$.

The definition (13) of $\varphi(t)$ shows that both $s\varphi(s)$ and $(h - s)\varphi(s)$ are majorized by $s(h - s)$ for $0 \leqq s \leqq h$. Hence (15) can be written as

$$h < \int_0^{t_0} s \mid g(s) \mid ds + \int_{t_0}^h (h - s) \mid g(s) \mid ds + \int_0^h s(h - s) \mid f(s) \mid ds. \tag{16}$$

If the sum S of the first two integrals is considered as a function of t_0, it is seen that its derivative is

$$dS/dt_0 = t_0 \mid g(t_0) \mid - (h - t_0) \mid g(t_0) \mid = (2t_0 - h) \mid g(t_0) \mid,$$

which is non-positive or non-negative according as $t_0 \leqq \frac{1}{2}h$ or $t_0 \geqq \frac{1}{2}h$. Hence the maximum of $S = S(t_0)$ for $0 \leqq t_0 \leqq h$ is attained either at $t_0 = 0$ or at $t_0 = h$. This fact, when combined with (16), leads to the criterion (7).

Ad (6). Since neither s nor $h - s$ exceeds h for $0 \leqq s \leqq h$ and since $s(h - s) \leqq (h/2)^2$, condition (6) is contained in (7).

Ad (5). In view of (4), the inequality (7) implies that

$$h < M_1 \max \left\{ \int_0^h s \, ds, \int_0^h (h - s) \, ds \right\} + M_2 \int_0^h s(h - s) \, ds.$$

Here the factor of M_1 is $h^2/2$ and that of M_2 is $h^3/6$. Hence (5) is contained in (7).

THE SIGN OF PLASTIC POWER IN THE GRAPHICAL TREATMENT OF PROBLEMS OF PLANE PLASTIC FLOW*

By WILLIAM PRAGER (*Brown University*)

1. Introduction. A graphical method of treating problems of plane plastic flow has been described in an earlier report, paper [1]**. This method uses the geometrical relations between the plane of flow (physical plane), the stress plane, and the hodograph plane. The coordinates of a point in the stress plane give the stress components σ_x and τ_{xy} at the corresponding point in the physical plane. The slip lines in the physical plane are mapped on to the cycloids in the stress plane that are described by the points of the Mohr circles as these roll without sliding on the lines $\tau_{xy} = \pm k$. The radius vector of a point in the hodograph plane gives the velocity at the corresponding point in the physical plane. A line element of a slip line in the physical plane is orthogonal to its images in the stress and hodograph planes. These relations are used to construct, mesh by mesh, the slip line net in the physical plane and its images in the stress and hodograph planes. Before, however, a slip line field obtained in this manner can be accepted as the solution to the considered problem, it must be shown that the plastic power is non-negative throughout this slip line field.

A method of investigating the sign of the plastic power for a tentative solution has been described by Green [2]. Since Green's criterion involves the difference of two expressions that depend on the radii of curvature of the slip lines, it is not always easy to apply. An alternative criterion, which is believed to be more convenient than Green's is given in this report.

2. Shear stresses acting on a mesh of the slip line net. Let the rectangle in Fig. 1a

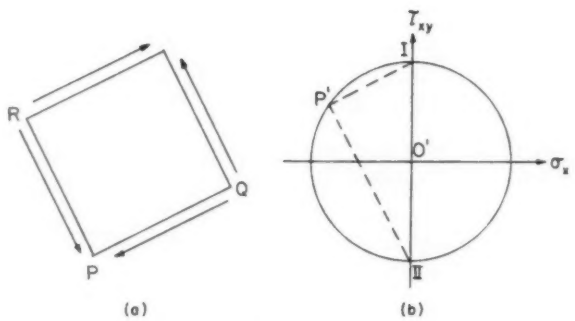

FIG. 1

represent a small mesh of the slip line net. It will always be possible to choose the corner P of this mesh so that the direction of the side PQ that falls on the first slip line through P and the direction of the other side PR through P form a right-handed system in the plane of flow. It will be assumed in the following that the point P has been chosen in this manner.

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**Numbers in square brackets refer to the Bibliography at the end of the report.

To determine the directions of the shear stresses transmitted on to the material inside the considered mesh, we use the Mohr circle of Fig. 1b. Since the superposition of a hydrostatic state of stress does not change the direction of the shearing stresses, we do not restrict the generality of our discussion by assuming the origin O' of the stress plane to coincide with the center of this circle. If the highest and lowest points of this circle are denoted by I and II, respectively, and if P' is the pole of this circle, the lines $P'I$ and $P'II$ are parallel to the shear lines PQ and PR, respectively. To find the direction of the shear stress transmitted across the side PQ on to the material inside the mesh, we must rotate the exterior normal to this side by a right angle in the same sense in which $O'I$ must be rotated by a right angle to coincide with the positive σ_x axis. This furnishes the shear stresses indicated in Fig. 1a. For the plastic power to be positive, the shear rate at P must therefore correspond to a *decrease* of the right angle at P.

3. Shear rate. As can be seen from Fig. 2, the rate at which the right angle at P

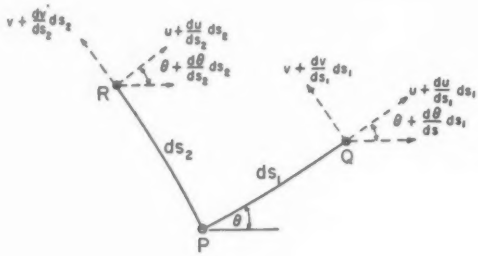

FIG. 2

is decreasing is given by

$$\gamma = \frac{dv}{ds_1} + u\frac{d\theta}{ds_1} + \frac{du}{ds_2} - v\frac{d\theta}{ds_2}, \tag{1}$$

where u and v are the components of the velocity vector in the first and second shear directions (PQ and PR), respectively, and θ is the angle between the first shear direction (PQ) and a fixed direction.

As is well known from the theory of the slip line field, the velocity components u and v satisfy the following conditions along the slip lines:

$$\frac{du}{ds_1} - v\frac{d\theta}{ds_1} = 0, \qquad \frac{dv}{ds_2} + u\frac{d\theta}{ds_2} = 0. \tag{2}$$

When the values of $d\theta/ds_1$ and $d\theta/ds_2$ obtained from (2) are substituted into (1), the following relation is obtained:

$$\gamma = \frac{dv}{ds_1} + \frac{u}{v}\frac{du}{ds_1} + \frac{du}{ds_2} + \frac{v}{u}\frac{dv}{ds_2}. \tag{3}$$

With $q^2 = \frac{1}{2}(u^2 + v^2)$, Eq. (3) can be written in the form

$$\gamma = \frac{1}{uv}\left[u\frac{dq}{ds_1} + v\frac{dq}{ds_2}\right]. \tag{4}$$

The bracket in Eq. (4) is positive, if the speed increases as one moves from P in the direction of the resulting velocity to a neighboring point P_1 on the stream line through P. If the images of the points P and P_1 in the hodograph plane are denoted by P'' and P_1'', the bracket in (4) is therefore positive, if P'' is closer to the origin O'' of the hodograph plane than P_1''. As the signs of u and v are readily determined, we have therefore a convenient criterion for the sign of the right-hand side of (4), that is for the sign of the plastic power. For a steady flow, in particular, this criterion requires that the particle that is instantaneously at P should be accelerating or decelerating according to whether u and v have the same signs or not.

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NECESSARY AND SUFFICIENT CONDITIONS FOR THE EXISTENCE OF NEGATIVE SPECTRA*

By C. R. PUTNAM (*Purdue University*)

Let $f(t)$ be a real-valued continuous function on the t-interval $0 \leqq t \leqq 2T$ and let $x = x(t)$ be a real-valued solution ($\not\equiv 0$) of the differential equation

$$x'' + f(t)x = 0. \tag{1}$$

If $f^+ = f$ or 0 according as $f \geqq 0$ or $f < 0$, and if $x(a) = x(b) = 0$, where $a < b$, then there holds the inequality

$$\int_a^b f^+(t)\,dt > 4/(b-a), \tag{2}$$

due essentially to Liapounoff (see [1]; also [8]). Moreover, according to [4] (see also [1]), the constant 4 of (2) is the best possible, in the sense that (2) need not hold (for arbitrary f) if the 4 is replaced by $4 + \epsilon$, where $\epsilon > 0$. Hence, it is easy to see that the inequality

$$T\int_0^T [f^+(t) + f^+(2T-t)]\,dt \qquad \left(= T\int_0^{2T} f^+(t)\,dt\right) > 2 \tag{3}$$

is necessary, but $T\int_0^{2T} f^+(t)\,dt > 2 + \epsilon$ is not, in order that the Sturm-Liouville boundary value problem

$$L(x) + \lambda x = 0 \qquad [L(x) \equiv x'' + fx], \qquad x(0) = x(2T) = 0 \tag{4}$$

possess an eigenvalue $\lambda < 0$ (or even $\leqq 0$).

It will be shown in this note that there is a *sufficient* criterion similar to (3). In fact,

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the following will be proved: *If $f(t)$ satisfies the inequality*

$$M(f) \equiv T^{-1} \int_0^T t^2[f(t) + f(2T - t)]\, dt > 2, \tag{5}$$

then the spectrum of the boundary value problem (4) *possesses some eigenvalue* $\lambda < 0$. Furthermore, the constant 2 occurring on the right side of (5) is the best possible in the sense that for every $\epsilon > 0$, there is a continuous function $f(t)$ satisfying $M(f) > 2 - \epsilon$, and such that the values λ in the spectrum of the eigenvalue problem (4) satisfy $\lambda > 0$. (Note that the similarity between (3) and (5) is particularly apparent if $f(t) \geqq 0$, so that $f = f^+$.)

If one grants, for the present, the italicized assertion, the proof of the claim following it is easy. In fact, let a continuous function $f(t)$ be defined on $0 \leqq t \leqq 2T$ so that $f(t) \geqq 0$ and so that

$$2 - \epsilon < M(f) < 2 - \epsilon/2 \qquad \text{and} \qquad \left| M(f) - T \int_0^{2T} f(t)\, dt \right| < \epsilon/2. \tag{6}$$

Clearly this can be done; one need only define f to be 0 everywhere on $0 \leqq t \leqq 2T$ except on some sufficiently small interval about T. Then the spectrum of the boundary value problem (5), for the above constructed f, consists of positive eigenvalues only. In fact, if there were an eigenvalue $\lambda \leqq 0$, there would exist a solution $x(t) \not\equiv 0$ of (1) satisfying $x(0) = x(T_1) = 0$, $0 < T_1 \leqq 2T$. Consequently, relation (2) for $a = 0$ and $b = T_1$ would now imply (3), that is $T \int_0^{2T} f(t)\, dt > 2$, in contradiction with (6).

There remains then to prove the italicized assertion. To this end, suppose, if possible, that there does not exist a negative eigenvalue, so that, in view of the Parseval equality, the relation

$$\int_0^{2T} (x'^2 - fx^2)\, dt = -\int_0^{2T} xL(x)\, dt = \sum \lambda_k c_k^2 \geqq 0 \tag{7}$$

holds for all real-valued functions x possessing continuous second derivatives on $0 \leqq t \leqq 2T$ and satisfying $x(0) = x(2T) = 0$. Here, the constants c_k are defined by $c_k = \int_0^{2T} x\phi_k\, dt$, where ϕ_k is the normalized eigenfunction belonging to the eigenvalue λ_k. (In connection with (7), see [3], p. 392 and the reference given there to Hamel [2]. For applications of an inequality similar to (7) but relating to singular boundary value problems [7] on the half-line $0 \leqq t < \infty$, see, e.g., [5], [6].) Next, define the continuous function $y(t)$ on $0 \leqq t \leqq 2T$ by putting $y(t) = t$ on $0 \leqq t \leqq T$ and $y(t) = 2T - t$ on $T \leqq t \leqq 2T$. (See [1], p. 69.) Then

$$\int_0^{2T} (y'^2 - fy^2)\, dt = 2T - \int_0^T t^2 f(t)\, dt - \int_T^{2T} (2T - t)^2 f(t)\, dt.$$

It follows from (5) that

$$\int_0^{2T} (y'^2 - fy^2)\, dt < 0.$$

Clearly, the function y can be approximated by functions x which possess continuous second derivatives on $0 \leqq t \leqq 2T$, satisfy $x(0) = x(2T) = 0$, and which violate the inequality of (7). This violation yields a contradiction and shows that the assumption that the spectrum contains no negative eigenvalues is untenable. The proof of the italicized assertion is thus complete.

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Correction to my paper

A NEW SINGULARITY OF TRANSONIC PLANE FLOWS*

Quarterly of Applied Mathematics, XII, 343-349 (1955)

BY A. R. MANWELL (*University College, Swansea*)

A much more detailed study of the singular solution discussed rather briefly in the note of the above title has shown that several statements in Sec. 4 are incorrect. Briefly, the expansions (4.4) and so also (4.5), (4.6) are valid only *locally* for either $\Theta = 0$ or for $\Theta = \pi$, but not necessarily for both. We may not infer from these expansions the existence of solutions in the whole interval $(0, \pi)$. (In particular, on account of the pole at $Z = 1$, we may not replace in (4.3) a contour for which $Z - 1 = 2i \exp(i\Theta) \sin \Theta$ is very small by the unit circle $Z = 1$).

A correct discussion shows that (4.3) and (4.4) yield only *two* independent solutions. As a consequence, the singular solution can be smoothly continued across the sonic line for $\Theta > 0$ but, unless we admit further singularities in the supersonic region, the flow would not join up smoothly for $\theta < 0$. Since we are seeking possible criteria for the breakdown of flow solutions, this correction leads to a slight strengthening of our original conclusion.

*Received March 7, 1955.

A MINIMUM PRINCIPLE OF PLASTICITY*

BY D. TRIFAN (*University of Arizona*)

This note is concerned with the removal of a certain restriction imposed by a proof[1] [Sec. 5] of a minimum principle of an isotropic, incompressible, strain-hardening material exhibiting a gradual transition from the elastic to the plastic state. The governing stress-strain relation for loading is given by

$$s_{ij}^* = 2G_0\epsilon_{ij}^* - p(E)\epsilon_{ij}E^*, \tag{1}$$

*Received Feb. 28, 1955.

[1]Section numbers enclosed in brackets refer to the following paper: D. Trifan, *A new theory of plastic flow*, Q. Appl. Math. **7**, pp. 201-211 (1949).

and for unloading by the differentiated form of Hooke's law

$$s_{ij}^* = 2G_0\epsilon_{ij}^* . \tag{2}$$

Variables with asterisks indicate rates of change with respect to any monotonically increasing function such as, for example, time. The stress deviation rate s_{ij}^* is defined in terms of the stress rate σ_{ij}^* by $s_{ij}^* = \sigma_{ij}^* - \frac{1}{3}\delta_{ij}\sigma_{kk}^*$—where repeated indices indicate summation in accordance with the summation convention of tensor calculus, and δ_{ij} the Kronecker delta—ϵ_{ij} is the strain tensor, $E = \frac{2}{3}\epsilon_{ij}\epsilon_{ij}$, G_0 is the shear modulus in the elastic range, and $p = p(E)$, a positive definite function depending on the material and satisfying the inequality $Ep(E) < G_0$ [Sec. 2].

Let a set of primed strain rates $\epsilon_{ij}^{*\prime}$ be called admissible strain rates if they are derivable from a set of velocities $u_i^{*\prime}$ by the relation $\epsilon_{ij}^{*\prime} = \frac{1}{2}(u_{i,j}^{*\prime} + u_{j,i}^{*\prime})$ where $u_{i,j}^{*\prime} = \partial u_i^{*\prime}/\partial x_j$, and the velocities in turn satisfy the incompressibility condition, given by $u_{i,i}^{*\prime} = 0$, and the boundary condition, given by surface velocities u_i^* , i.e., $u_i^{*\prime} = u_i^*$ on the surface. The corresponding admissible stress deviation rates $s_{ij}^{*\prime}$ can be determined, for a given state ϵ_{ij} throughout the body, by Eq. (1) when $E^{*\prime} = \frac{4}{3}\epsilon_{ij}\epsilon_{ij}^{*\prime} > 0$, and by Eq. (2) when unloading occurs, $E^{*\prime} < 0$ [Sec. 2]. The actual strain rate ϵ_{ij}^* existing in the body is an admissible strain rate whose corresponding stress rate σ_{ij}^* satisfies the equations of equilibrium $\sigma_{ij,j}^* = 0$.

MINIMUM PRINCIPLE. For any admissible set of strain rates and corresponding stress rates, the integral $J(\epsilon_{ij}^{*\prime}) = \int_V \epsilon_{ij}^{*\prime}\sigma_{ij}^{*\prime}\, dv$ is an absolute minimum for the actual strain and stress rates occurring in the deformed body, i.e.,

$$J(\epsilon_{ij}^{*\prime}) - J(\epsilon_{ij}^*) \geq 0.$$

Equality holds only if $\epsilon_{ij}^{*\prime} = \epsilon_{ij}^*$.

Proof. Due to the incompressibility condition $\epsilon_{ii}^{*\prime} = 0$, the following contracted tensor products are equivalent $\epsilon_{ij}^{*\prime}\sigma_{ij}^{*\prime} = \epsilon_{ij}^{*\prime}s_{ij}^{*\prime}$, and thus by Eqs. (1) and (2) the integrals $J(\epsilon_{ij}^{*\prime})$ and $J(\epsilon_{ij}^*)$ become

$$J(\epsilon_{ij}^{*\prime}) = 2G_0\int_V \epsilon_{ij}^{*\prime}\epsilon_{ij}^{*\prime}\, dv - \frac{3}{4}\int_{V_+'} p(E)\{E^{*\prime}\}^2\, dv, \tag{3}$$

$$J(\epsilon_{ij}^*) = 2G_0\int_V \epsilon_{ij}^*\epsilon_{ij}^*\, dv - \frac{3}{4}\int_{V_+} p(E)\{E^*\}^2\, dv. \tag{4}$$

The sub-domains of volume V where loading and unloading occur are indicated by plus and minus subscripts respectively. In general the actual sub-domains V_+ and V_- are not known; however, for admissible strain rates V_+' indicates the sub-domain where $E^{*\prime} > 0$, and V_-' the sub-domain where $E^{*\prime} < 0$. It is evident that the second integrals in (3) and (4) are to be integrated over the loading domains only. It is necessary to consider the portion of the volume where actual loading occurs while admissible strains indicate unloading; this we denote by V_{+-} . The sub-domain V_{-+} indicates actual unloading while admissible strains assume loading.

It is convenient at this point to introduce the following notation[2]:

$$u_i^{*\prime} = (u_i^{*\prime} - u_i^*) + u_i^* = \Delta u_i^* + u_i^* ,$$
$$\epsilon_{ij}^{*\prime} = (\epsilon_{ij}^{*\prime} - \epsilon_{ij}^*) + \epsilon_{ij}^* = \Delta\epsilon_{ij}^* + \epsilon_{ij}^* ,$$
$$E^{*\prime} = (E^{*\prime} - E^*) + E^* = \Delta E^* + E^*,$$
$$V_+' = V_+ + V_{-+} - V_{+-} .$$

[2]The author is indebted to H. J. Greenberg for this suggestion.

Integral (3) now becomes

$$J(\epsilon_{ij}^{*\prime}) = 2G_0 \int_V (\Delta\epsilon_{ij}^{*}\Delta\epsilon_{ij}^{*} + 2\Delta\epsilon_{ij}^{*}\epsilon_{ij}^{*} + \epsilon_{ij}^{*}\epsilon_{ij}^{*})\, dv$$

$$- \frac{3}{4}\int_{V_+} (\{\Delta E^*\}^2 + 2\Delta E^* E^* + \{E^*\}^2)p(E)\, dv$$

$$- \frac{3}{4}\int_{V_{-+}} \{E^{*\prime}\}^2 p(E)\, dv + \frac{3}{4}\int_{V_{+-}} \{E^{*\prime}\}^2 p(E)\, dv. \qquad (5)$$

Since

$$\int_V s_{ij}^{*}\Delta\epsilon_{ij}^{*}\, dv = \int_V \sigma_{ij}^{*}\Delta\epsilon_{ij}^{*}\, dv = \int_V \sigma_{ij}^{*}(\Delta u_i^*)_{,j}\, dv = \int_V (\sigma_{ij}^{*}\Delta u_i^*)_{,j}\, dv$$

$$= \int_S \sigma_{ij}^{*}\Delta u_i^* n_j\, dS = 0,$$

due to the incompressibility condition, strain rate—velocity relation, symmetry of the stress rates, equations of equilibrium, divergence theorem, and the fact that $\Delta u_i^* = 0$ on the surface, it follows by Eqs. (1) and (2) that

$$2G_0 \int_V \epsilon_{ij}^{*}\Delta\epsilon_{ij}^{*}\, dv - \frac{3}{4}\int_{V_+} E^*\Delta E^* p(E)\, dv = 0. \qquad (6)$$

A simplification of integral (5) is possible with the application of (4) and (6)

$$J(\epsilon_{ij}^{*\prime}) - J(\epsilon_{ij}^{*}) = 2G_0 \int_V \Delta\epsilon_{ij}^{*}\Delta\epsilon_{ij}^{*}\, dv - \frac{3}{4}\int_{V_+} \{\Delta E^*\}^2 p(E)\, dv$$

$$- \frac{3}{4}\int_{V_{-+}} \{E^{*\prime}\}^2 p(E)\, dv + \frac{3}{4}\int_{V_{+-}} \{E^{*\prime}\}^2 p(E)\, dv. \qquad (7)$$

In V_{-+} it is evident that $\{\Delta E^*\}^2 > \{E^{*\prime}\}^2$, and since V_{-+} is included in V_-,

$$\int_{V_{-+}} \{E^{*\prime}\}^2 p(E)\, dv < \int_{V_{-+}} \{\Delta E^*\}^2 p(E)\, dv < \int_{V_-} \{\Delta E^*\}^2 p(E)\, dv.$$

This inequality applied to (7) in conjunction with the positive character of the integral over V_{+-} reduces (7) to

$$J(\epsilon_{ij}^{*\prime}) - J(\epsilon_{ij}^{*}) > 2G_0 \int_V \Delta\epsilon_{ij}^{*}\Delta\epsilon_{ij}^{*}\, dv - \frac{3}{4}\int_V \{\Delta E^*\}^2 p(E)\, dv. \qquad (8)$$

By means of the Schwarzian inequality

$$\{\Delta E^*\}^2 = \left\{\frac{4}{3}\epsilon_{ij}\Delta\epsilon_{ij}^{*}\right\}^2 \leq \frac{16}{9}\epsilon_{ij}\epsilon_{ij}\Delta\epsilon_{rs}^{*}\Delta\epsilon_{rs}^{*} = \frac{8}{3}E\Delta\epsilon_{ij}^{*}\Delta\epsilon_{ij}^{*},$$

so that

$$J(\epsilon_{ij}^{*\prime}) - J(\epsilon_{ij}^{*}) > 2\int_V \{G_0 - Ep(E)\}\Delta\epsilon_{ij}^{*}\Delta\epsilon_{ij}^{*} > 0.^3$$

It has been assumed in this proof that $\epsilon_{ij}^{*\prime} \neq \epsilon_{ij}^{*}$. The case when $\epsilon_{ij}^{*\prime} = \epsilon_{ij}^{*}$ is trivial and leads to the equality in the minimum principle.

[3] In [Sec. 5] it was necessary to assume that $E^{*\prime} > 0$ whenever $E^* > 0$, i.e. no V_{+-} domain, in order to obtain this result.

BOOK REVIEWS

(Continued from p. 270)

Selected papers in statistics and probability. By Abraham Wald. McGraw-Hill Book Co., New York, Toronto, London, 1955. ix + 702 pp. $8.00.

This book contains reproductions of most papers written by the late Abraham Wald on the subject of statistics and probability except those dealing with topics already included in other books written by Wald. It also contains a short biography, a section describing some of the works of Wald particularly in regard to their connection with the works of other authors, a chronological listing of all his publications and one paper that has not previously been published. The book is a fitting memorial to an exceedingly productive mathematician.

GORDON NEWELL

A short table for Bessel functions of integer orders and large arguments. Prepared on behalf of the Mathematical Tables Committee of The Royal Society by L. Fox. University Press, Cambridge, 1954. 28 pp. $1.25.

Tables for computation of the functions $J_n(x)$, $Y_n(x)$, $I_n(x)$, $K_n(x)$ for integral values of n from 0 to 20 and for values of x covering the range $20 \leq x \leq \infty$. The tabulation gives the numerical values of auxiliary functions in terms of the auxiliary argument $1/x$. These can now be used to readily compute the required Bessel functions by means of their asymptotic expansions.

G. W. MORGAN

Decision processes. Edited by R. M. Thrall, C. H. Coombs, R. L. Davis. John Wiley & Sons, Inc., New York, Chapman & Hall, Ltd., London, 1954. viii + 332 pp. $5.00.

This collection of papers arises from a seminar, held in the summer of 1952, in which a group of "mathematicians, statisticians, psychologists, economists, and philosophers" studied various aspects of "decision processes." The range of topics treated in this seminar is given by the headings of the four groupings under which the papers are published: Individual and Social Choice, Learning Theory, Theory and Applications of Utility, Experimental Studies. The approaches used include an exposition of some of the elements of real variables, the comparison of stochastic and game theoretic models for individual and group behavior against experimental evidence, the development of stochastic models for learning and the comparison of criteria for making decisions.

Briefly, the purpose of most of the authors seems to be to go beyond the purely elementary descriptive approach, e.g., systems of linear equations, etc., usually found in the social sciences and develop more subtle models for the relationships between the various aspects of observed or potentially observable systems. The problem of bridging the gap between observation and model in the sense of Wald's statistical decision theory is treated only incidentally.

The papers are generally at a high and thoughtful level but the reviewer noticed no new results which appeared to be important from a purely mathematical point of view. Those whose work brings them in contact with theoretically inclined social scientists will, however, find this book useful in giving them a feel for the state of knowledge in learning and organization theory and a starting point for their own research.

ISADORE BLUMEN

Mathematics in type. The William Byrd Press, Inc., Richmond, Virginia, 1954. xii + 58 pp. $3.00 (50% discount to staff members of educational institutions).

Most mathematicians and scientists are aware of the appalling cost of mathematical composition, but few have sufficient insight into the mechanics of modern typesetting to realize how much of this

cost is caused by the use of typographically inconvenient notation. The same mathematical idea can, in general, be expressed by any one of a number of systems of notation. All too often an author's choice of one of these is based on considerations involving the ease of writing rather than that of typesetting. Thus, "economic forces are set in motion which tend to limit the ultimate market for books and to reduce the number of pages published in journals". This situation is not likely to improve until authors of mathematical papers and books acquire a rudimentary knowledge of the mechanics of setting mathematical type. The present booklet contains a clear discussion of the difficulties of monotype composition of mathematical formulas; it should be required reading for all authors using more than the most elementary mathematical symbols.

W. PRAGER

Optics. By Arnold Sommerfeld. Academic Press, Inc., New York, 1954. xi + 383 pp. $6.80.

This is an English translation by O. Laporte and P. A. Moldauer of Volume IV of Lectures on Theoretical Physics by Sommerfeld. This text on physical optics is divided into six chapters, the first of which is concerned with the reflection and refraction of light and includes the usual treatment of the Fresnel formula, total reflection and metallic reflection. The first chapter also includes an unusually concise treatment of reflection and refraction and interference behavior in thin films and thick plates including the Lummer-Gehrke plate, the Fabry-Perot interferometer. Chapter two is concerned with the optics of moving media and sources. In particular, the Doppler effect, the Michaelson experiment, reflection by a moving mirror, and the effect of the earth's rotation on the propagation of light are some of the topics treated. Chapter three is concerned with the theory of dispersion. This chapter contains the main features of classical theory including discussion of magnetic rotation of polarization plane and the normal Zeeman effect. Phase velocity, signal velocity, group velocity and energy transport are discussed. An outline of the wave-mechanical theory of dispersion is given. Chapter four deals with crystal optics and develops from Maxwell's equations the plane wave behavior in optical crystals. The usual ray surface and wave normal surfaces are found together with their relation to the optic axes. The optical symmetry of crystals is discussed, as well as the rotation of the plane of polarization of a wave propagation. Chapter five and much of chapter six are concerned with the theory of diffraction. In addition to a rather complete discussion of what might be called pre-war diffraction theory, there is in chapter six some discussion of the newer theory of Bethe and Levine and Schwinger. Discussion of a reformulation of Kirchoff's solution of the diffraction problem occupies part of chapter six. There is a very interesting section in Cerenkov radiation where the idea of a Mach cone for a Cerenkov electron is developed. The electromagnetic field is zero outside the Mach cone but exists everywhere in the interior of the cone. When dispersion is taken into account, the difference between the phase and group velocity affects the size of the Mach cone.

Finally, there is a section in chapter six on geometrical optics and the existence of the eikonal. Discussion of curved light rays and rectilinear ray bundles (lens formula) is also included.

It is undoubtedly superfluous to say that this is one of the better texts on physical optics. Looking for mistakes or careless work in something written by Sommerfeld is like looking for a bad meal in France. This book is not a treatise such as Born's or Fosterling's or Volume 20 of the Handbuch der Physik; it is an up-to-date combination of intermediate and advanced textbook on physical optics.

ROHN TRUELL

Nomographie. By M. W. Pentkowski. Akademie Verlag, Berlin, 1953. xv + 268 pp. $3.58.

In refreshing contrast to other recent texts in this field, nearly one half of this book is devoted to the theoretical foundations of nomography. The first two chapters treat alignment charts involving three variables. Following Soreau, the concept of the nomographic order is used as a convenient basis for the systematic discussion of these charts. Subsequent chapters deal with network charts, alignment

charts containing networks, composite charts, and special methods of nomographic representation. The second half of the work is devoted to practical methods of constructing various types of charts, special attention being given to the projective transformation of charts. The clear exposition of nomographic theory and practice is supported by numerous skillfully drawn and well-reproduced figures.

W. PRAGER

Economic activity analysis. Edited by O. Morgenstern. John Wiley & Sons, New York, 1954. xviii + 554 pp. $6.75.

The thirteen papers of the volume are grouped in three parts labelled Economic Properties of the Input-Output System, Mathematical Properties of Linear Economic Systems, and Meta-Economics. Part I contains two expository papers on models of general economic equilibrium (by J. Balderston) and on the input-output system (by O. Eckstein), which will be particularly welcome to the applied mathematician unfamiliar with economic theory. The remaining papers of Part I are concerned with the role of aggregation in economic models and the treatment of foreign and domestic trade and transportation charges. Part II consists of seven papers of primarily mathematical contents; most of these are concerned with properties of certain types of matrices that occur in linear models of economic systems. For example, the elements of a Minkowski-Leontieff matrix are non-negative, and the sum of the elements in any column cannot exceed unity. Part III contains a translation of slightly revised versions of two papers by K. Menger on the logic of the laws of return. These were published in Vienna in 1936, but have been largely ignored in the English-American literature on economics. A final paper by O. Morgenstern is concerned with experiment and large-scale computation in economics.

W. PRAGER

Gas dynamics of thin bodies. By F. I. Frankl and E. A. Karpovich. Interscience Publishers, Inc., New York and London, 1954. viii + 175 pp. $5.75.

This monograph, which is a translation of a Russian text first published in 1948, presents a mathematical treatment of the linearized theory of compressible flow of a perfect, inviscid, and non-heat conducting fluid. Throughout the work only irrotational flows are considered so that the presence of a velocity potential can be assumed. Both subsonic and supersonic steady as well as unsteady flows are discussed.

The book is divided into five chapters, the first of which contains a rather nationalistic historical survey of the subject, along with a formulation of the three-dimensional unsteady problem in terms of the wave equation. The idea of the retarded potential (supersonic source) as the fundamental solution is introduced. Chapter II considers the solution to the steady compressible subsonic and supersonic flow around slender bodies of revolution by the use of the axially distributed source, and in the case of angle of attack, the distributed doublet. The unsymmetrical body which is not too different from a body of revolution, as well as the case of axially accelerated motion, are also briefly treated. The third chapter is concerned with the steady motion of a wing, and in it is to be found a review of the problem of the infinite span wing at subsonic and supersonic speeds, as well as the lifting line theory for finite span wings in subsonic flow. The remainder of the chapter is devoted to finite span wings in supersonic flow. The general drag and lift problem for a wing of zero thickness in supersonic flow is solved by the method of reducing the solution to a sequence of Abel integral equations. After this general treatment, the acceleration potential is applied to the problem of the lifting trapezoidal wing at supersonic speeds. Finally, the concept of the supersonic source is utilized to discuss the flow around symmetric wings with thickness and large sweep. Chapter IV examines the problem of the unsteady motion of airfoils and wings executing small harmonic oscillations. A solution for the oscillating airfoil in subsonic flow is given, and by the introduction of the pulsating supersonic source, the two- and three- dimensional problem at supersonic speeds is also considered. The remainder of this chapter is devoted to an account

of the linearized theory of high speed propellers based on the lifting line theory. In the final chapter the linearized theory of conical flow is presented along with its use in supersonic wing theory through the application of conformal mapping techniques. Some attention is also given to the generalized linear conical flow theories.

This reviewer cannot, for several reasons, agree with the translator's opinion that this ". . . should certainly make a fine text book for a one-year course in linearized supersonic aerodynamics". The primary reason stems from the nature of the approach which is purely mathematical, and which is characterized by an almost complete lack of stated physical assumptions and arguments, a characteristic evident in most Soviet papers on the subject. This trait is exemplified in the discussion of wave drag on p. 29 where a Mach wave (cone) is indicated as being the same as a shock wave. Of course, the second feature that detracts from its use as a text is the nationalistic treatment which results in the exclusion of the names and work of such people as Glauert, Jones, Lighthill, Evvard, Lagerstrom, Stewart, and Ward, to name but a few. As noted by the translator, numerous misprints are prevalent in Soviet texts; however, although an attempt has in this instance been made to correct them, it would appear that a significant number still remain. Finally, because the translation is literal much is left to be desired in the way of readability for those familiar, or those first becoming acquainted with aerodynamic terminology; e.g. the use of "compression layers" for shock waves, "elongated body" for slender body, "wave resistance" for wave drag, etc.

In spite of the preceding criticism, it is this reviewer's opinion that this monograph should prove useful as a general reference work, or as a supplementary text in a course in which the subject matter is covered. It should also prove worthwhile for those workers or students who have some familiarity with the field, and who desire a unified mathematical approach to the topic.

RONALD F. PROBSTEIN

The dynamics and thermodynamics of compressible fluid flow. By Asher H. Shapiro. Volume II. The Ronald Press Co., New York, 1954. xi + 535 pp. Volume II: $16.00 (the set $30.00).

This is the second volume written by Professor Shapiro on compressible fluid flow theory. The first volume has been reviewed in a former issue of this *Quarterly*. Thus, the general characteristics of the work have been emphasized previously and the present review can be restricted to the contents of the second volume.

To begin with, the author continues (Part V) the treatment of supersonic flow theory. Three chapters (Axially symmetric supersonic flow, Supersonic flow past wings of finite span, Hypersonic flow) are devoted to this material. Some parts are treated in great detail, others—especially the general method used in linearized theory—are only outlined. The chapter on hypersonic flow will be found useful because the material given there does not appear in the classical books. The next part is devoted to mixed flow, i.e. flow which is partly subsonic and partly supersonic. The first chapter is a very classical—source, vortex, compressible flow with 180° turn are treated by the hodograph method—special attention (perhaps too much) is given to the limiting line. Transonic flow is then considered in the physical plane with special emphasis on similarity laws and applications to the flows through a converging-diverging nozzle and along a wavy wall. A third chapter is concerned with experimental results at transonic speeds and their interpretation; drag, lift, occurrence of shock waves and their interactions with the boundary layer are discussed.

Unsteady motion in one dimension is treated in Part VII. The first chapter deals with linearized theory (acoustical approximation). This provides a good introduction to the more difficult case arising when the disturbances are not small (second chapter); in this case the method of characteristics finds a very elegant application. Special attention is given to the effects of area change, friction, and entropy variations. Specific examples and comparisons of experiments with the theory are presented. Finally, the theory of unsteady one-dimensional shock waves is discussed, in particular the interactions of shocks with each other and with continuous waves is considered. The final part of this work is concerned with flow of real gases, taking account of viscosity and heat conductivity; only the boundary layer effects are considered. In the first chapter the fundamental concepts together with the corresponding differentia.

and integral equations (Momentum and Energy) are introduced; applications are then made to the laminar boundary layer. The next chapter deals with the corresponding problems associated with the theory of the turbulent boundary layer, which is perhaps of greater practical significance but far more difficult to understand. The analysis is based on the so-called "Reynolds stresses". A great deal of significant experimental information is found at the end of this chapter. Finally, a special chapter is devoted to boundary layers in tubes, in particular in the presence of shock waves.

As in the first volume, the material covered in this book is very extensive. Much valuable information in the form of introductions to many theories and their developments, summarization and discussion of various experiments, various tables and charts make this work very useful for the aeronautical engineer who must apply the results of Gas Dynamics to practical problems. It is perhaps not so advisable however to recommend it to a student who has not received a sound basic training or read previously some more elementary book in which the fundamental ideas are introduced. But for the manifold data which can be found in these two books, we must be very grateful to the author.

P. GERMAIN

A selection of graphs for use in calculations of compressible airflow. Prepared on behalf of the Aeronautical Research Council by the Compressible Flow Tables Panel (L. Rosenhead, Chairman, W. G. Bickley, C. W. Jones, L. F. Nicholson, H. H. Pearcey, C. K. Thornhill, R. C. Tomlinson). Oxford University Press, 1954. x + 115 pp. $13.45.

This book, which is a companion volume to "Compressible Airflow: Tables", makes available to engineers, physicists, and applied mathematicians a set of graphs which is useful for calculations involving the steady compressible flow of air for Mach numbers not greater than five. There are five sections to the book: (A) isentropic flow, both subsonic and supersonic; (B) normal shocks; (C) oblique shocks, including shock reflection from a plane wall; (D) conical flow, which treats the axisymmetric, supersonic flow past a cone with an attached conical shock at its vertex; (E) the determination of Reynolds number from specified values of Mach number, stagnation pressure, and stagnation temperature, or specified values of Mach number and altitude. The graphs of each section are preceded by a relatively short descriptive passage on the theory involved, in which the plotted equations are clearly indicated. The graphs themselves, which are large (size 8″ x 11″) and easily read, are of two types: single-page graphs, which are suitable for rough calculations; and multiple-page graphs, in which the single-page graphs are plotted to a much larger scale. The table of contents indicates the range of abscissa and ordinate and the values of the parameters used for each graph. Aside from the fact that the section on conical flow is particularly useful, the most important feature of this book is that it presents data graphically. Such a method of presentation makes it possible to consider either variable as being the dependent one and eliminates the necessity of making numerical interpolations.

P. F. MAEDER

Mathematics and plausible reasoning. Volume I: Induction and analogy. Volume II: Patterns of plausible inference. By G. Polya. Princeton University Press, 1954. Vol. I: xvi + 280 pp. $5.50. Vol. II: x + 190 pp. $4.50. (The set: $9.00).

It does not seem possible to convey a reasonably complete idea of the contents of these volumes within the space available for this review; the most that the reviewer can hope to accomplish is to induce many readers of this review to discover for themselves the attractions of this charming work. *Plausible reasoning* is a most important tool of the creative mathematician. In publishing his results, however, the mathematician is apt to hide the beautiful grain of plausible reasoning under a glossy lacquer of *demonstrative reasoning*. While this policy may often be dictated by limitations on space, there is no excuse for its adoption in teaching. "If the learning of mathematics reflects to any degree the invention of mathematics, it must have a place for guessing, for plausible inference." "A serious student of mathematics, intending to make it his life's work, must learn demonstrative reasoning; it is his profession and the distinctive mark of his science. Yet for real success he must also learn plausible reasoning; this is the kind of

reasoning on which his creative work will depend." The author warns the reader that "the efficient use of plausible reasoning is a practical skill and is learned, as any other practical skill, by imitation and practice." The first volume contains carefully chosen examples of plausible reasoning drawn from many branches of pure and applied mathematics. In many cases the author must content himself with offering conjectures regarding the steps that led to a discovery; in others he knows what happened either because the examples are drawn from his own work or because the discoverer left sufficient clues for the reconstruction of his chain of thought. Euler's work is particularly noteworthy in this respect because "he takes pains to present the relevant inductive evidence carefully, in detail, in good order," adding "to the discoveries, with which he enriched science, the candid exposition of the ideas that led him to those discoveries" (Condorcet). Chapters on plausible reasoning in different branches of mathematics intersperse chapters on methodology. Each chapter is followed by comments and examples. The latter provide the reader with a welcome opportunity to test his skill; they often form suggestive sequences. In the second, more philosophical, volume, the author undertakes to formulate certain patterns of plausible reasoning and discuss their relation to the Calculus of Probability. The final chapter is devoted to the role of plausible reasoning in invention and instruction. Throughout the work, simplicity seems to have governed the author's choice of examples, but he has by no means overlooked mathematical beauty or historical interest. Altogether, this is a stimulating and challenging work, which, it is hoped, will in time profoundly influence the teaching of mathematics.

W. PRAGER

Methods of mathematical physics. Vol. I. By R. Courant and D. Hilbert. Interscience Publishers, New York, 1953. xv + 561 pp. $9.50.

It is indeed a pleasure to have this standard text and reference work available at last in the English language. The translation from the 1931 German original was done by Professor Courant himself, and is entirely clear and readable.

The revisions announced on the title page are not major ones. A short section on the quasi-geometric interpretation of reciprocal quadratic variational problems has been added to the chapter on the calculus of variations, and the discussion of Castigliano's principle in the theory of elasticity has been altered to make use of this. An appendix to the last chapter discusses the transformation of spherical harmonics under rotation of coordinate axes. The references have been brought up to date by the inclusion of a supplementary bibliography at the end of the book.

STEPHEN PRAGER

Superfluids, macroscopic theory of superfluid helium. By Fritz London. Vol. II. John Wiley & Sons, Inc., New York, and Chapman & Hall, Ltd., London, 1954. xvi + 217 pp. $8.00.

This second volume on the subject of superfluids represents the last major work of a most distinguished scientist. Despite the designation as Volume II, the book is self-contained and is concerned exclusively with properties of liquid helium. Only occasionally are references made, by way of analogy, to the subject matter of Volume I, superconductivity.

Even though the theory of liquid helium still contains numerous controversies and is still developing at a very rapid rate, most of the basic concepts discussed in this book are expected to remain valid. The book is concerned primarily with the theory although comparisons with experimental data are very plentiful. Despite the fact that the author originated and has been one of the chief promoters of the interpretation of the λ-point as being due to a Bose-Einstein condensation, the various other conflicting theories are also given careful consideration. The major part of the book deals, however, with the macroscopic rather than the microscopic properties of the superfluid state.

The book certainly gives a very nice survey of literature up to early 1953 with emphasis on physical principles rather than just mathematical expressions and adequately fills a most conspicuous vacancy that has existed in the scientific literature for many years.

G. F. NEWELL

Heat conduction with engineering, geological, and other applications. By L. R. Ingersoll, O. J. Zobel, and A. C. Ingersoll. Revised edition. The University of Wisconsin, Madison, 1954. xiii + 325 pp. $5.00.

The present book is a revised edition of the well known book on the 'Mathematical Theory of Heat Conduction', first published more than 40 years ago, and later expanded to nearly its present size. The last edition includes two new chapters: Chap. 13 entitled "Theory of earth heat exchangers for the heat pump", and Chap. 14 entitled "Drying. Soil consolidation".

The original aim of the authors was "to develop the subject with special reference to the needs of the student who has neither time nor mathematical preparation to pursue the study at great length . . . [and] . . . to point out the many applications of which the results are susceptible." The authors have succeeded in achieving this aim, particularly in pointing out the numerous problems arising in geology, and in mechanical, metallurgical and civil engineering which require the calculation of heat conduction. The applications discussed in the book range from thawing of frozen soil, cooling of the earth, cooling of lava under water to the removal of shrink fittings, temperature fluctuations in engine cylinders, ground pipe heat sources and others, too numerous to quote here in detail.

No attempt is made to classify the topics into their appropriate fields of application, each topic being discussed to illustrate problems as they arise in their natural sequence.

The treatment is simple and includes remarks on such mathematical topics as Fourier series, a short table of formulae, the use of conjugate functions for the solution of Laplace's equation, short tables of definite and indefinite integrals and several useful functions.

In limiting themselves to elementary mathematical tools, the authors exclude the much more powerful modern methods of handling the heat conduction equation, but they include a useful short discussion of auxiliary methods of treating heat-conduction problems (electrical methods, the use of tables and curves, the Schmidt method, the relaxation method and the step method). In all cases the essentials are explained in a simple manner, and no deeper understanding is sought.

The more specialized reader will find the scope of the work too narrow. On the one hand the applied mathematician will miss the more advanced and more powerful methods of analysis and will probably prefer to turn to Carslaw and Jaeger's classical treatise on the subject ('The conduction of heat in solids', Clarendon Press, Oxford, 1947). The mechanical engineer will acquire an appreciation of the variety of possible applications. He will, however, find that the number of aids in the form of graphs, charts, and tables is inadequate. He will also have to turn to the original references on Schmidt's graphical method or on relaxation methods, before he can make use of them. He will, most probably, prefer to make use of the comprehensive volume by Max Jakob ('Heat transfer', vol. I, John Wiley, New York, 1949).

The book is very well laid out, printed in large clear type, and provided with an adequate Index and with a fairly extensive, but rather selective list of references. In this connection it may be pointed out that Schmidt's second paper on graphical methods (Forschung **13**, 177-185, 1942) seems to have escaped the authors' notice. In describing the work on thermal conductivity the very important contributions due to Keyes (e.g. Trans. ASME **76**, 809, 1954) find no mention, either.

Several small omissions and incongruities might usefully be corrected in the future. The present reviewer prefers the designation "error function" to "probability integral" used by the authors and finds that the Bessel functions (of the first kind) used are not adequately described. He also deplores the use of the symbols fph and cgs to designate indiscriminately individual units in either of the two systems of units.

Within the limitations which the authors have set themselves, they have succeeded in writing a useful first introduction to the subject.

J. Kestin

Dynamical theory of crystal lattices. By Max Born and Kun Huang. Oxford at the Clarendon Press, 1954. xii + 420 pp. $8.00.

This book was written to satisfy a need for a newer, more comprehensive treatise on lattice dynamics and crystal theory. The first three chapters primarily by K. Huang constitute Part I of this book and

they are concerned with atomic forces, lattice vibrations, and elasticity and mechanical stability (of lattices). The last four chapters are due primarily to M. Born and they are entitled Quantum Mechanical Foundation, The Method of Long Waves, The Free Energy, and The Optical Effects.

The text is one on the mechanics of the lattice and crystal dynamics; it is not a text on the general theory of solids (or metals), nor does this book contain any discussion of the diffraction of X-rays, electrons or neutrons by crystal lattices. The interaction of lattice vibrations and scattered particles is also omitted "only with regret". There is no treatment of "defects" of the crystal lattice.

Chapter six is concerned with the thermodynamics of lattices and most parts of it appear to be new. This chapter involves discussion of the elasticity of lattices under finite strains. There is detailed analysis of the elastic, piezoelectric, and pyroelectric properties of dielectrics, including the temperature dependence of these properties.

The first three chapters of this book are considerably more descriptive and physical in nature than the last four chapters which are highly mathematical.

While the material in this book deals mainly with the lattice mechanics of non-conducting materials, the material is obviously fundamental to the whole theory of solids, and it appears to be a very valuable contribution to the more important texts in this field.

ROHN TRUELL

Formulas for computing capacitance and inductance. By Chester Snow. National Bureau of Standards Circular 544, issued September 1954. 69 pp. $.40.

This is a collection of explicit formulas for the calculation of (1) the capacitance between conductors having a large variety of geometrical configurations, (2) the inductance (self and mutual) of circuits of various geometries, (3) forces acting between current-carrying coils. Formulas for skin effect and other frequency effects are included. The resulting expressions involve elementary functions, Legendre polynomials, Legendre functions, and elliptic functions.

ROHN TRUELL

Multipole fields. By M. E. Rose. John Wiley & Sons, Inc., New York, Chapman & Hall, Ltd., London, 1955. viii + 99 pp. $4.95.

This book is concerned with the angular momentum and transformation properties of multipole electromagnetic fields in physical systems, especially nuclei. Internal conversion, the emission and absorption of electromagnetic radiation, the emission of β particles, and the static interactions of nuclear moments with fields caused by ions or other charges, are examples dealt with in this book—examples where the angular momentum properties are important.

The chapter headings in the text are: I. The Classical Field Equations. II. The Multipole Fields. III. Properties of the Multipole Fields. IV. The Retarded Electromagnetic Interaction. V. Internal Conversion. VI. Emission of Gamma Radiation.

ROHN TRUELL

Quantum mechanics. By F. Mandl. Academic Press, New York, Butterworth's Scientific Publications, London, 1954. viii + 233 pp. $5.80.

This book arose as the result of a course of lectures on quantum mechanics, primarily for experimentalists, at the Atomic Energy Research Establishment at Harwell in 1952. The first five chapters are used to develop the machinery of quantum mechanics in fairly concise fashion. Chapter six discusses in some details the angular momentum, spin, and symmetry properties of systems of particles. Chapter

eight deals in a very clear way with an introduction to scattering problems. Chapter nine is concerned with an introduction to the ideas of group theory.

One of the attractive features of the book, in addition to its conciseness, is a rather large collection of exercises and problems.

ROHN TRUELL

Numerische Behandlung von Differentialgleichungen. By L. Collatz. Second edition. Springer-Verlag, Berlin, Göttingen, Heidelberg, 1955. xv + 526 pp. $8.65.

The first edition, which appeared in 1951, was reviewed in this Quarterly, vol. 12, pp. 92. The present volume exceeds the earlier one by 68 pages. The principal change consists in the addition of an introductory chapter of 45 pages, which is devoted to background material and general principles. References to this chapter frequently enable the author to achieve a more concise presentation in subsequent chapters.

The smaller changes and additions are too numerous to be listed here; they are mostly concerned with extensions of the theory, in particular improvements in the analysis of errors.

W. PRAGER

Engineering cybernetics. By H. S. Tsien. McGraw-Hill Book Co., Inc., New York, Toronto, London, 1954. xii + 289 pp. $6.50.

Cybernetics is defined by the author in his Preface as "the science of organization of mechanical and electrical components for stability and purposeful actions." The author distinguishes the science of engineering cybernetics from the practice of servomechanisms engineering, and aims at making "a comprehensive survey of the whole field" of the new science.

A particular sub-field of engineering cybernetics is examined in each one of the 18 chapters, and the best way to show the unusually broad coverage of this work is to list the chapter titles, which follow: 1. Introduction 2. Method of LaPlace Transform 3. Input, Output, and Transfer Function 4. Feedback Servomechanism 5. Noninteracting Controls 6. Alternating-current Servomechanisms and Oscillating Control Servomechanisms 7. Sampling Servomechanisms 8. Linear Systems with Time Lag 9. Linear Systems with Stationary Random Inputs 10. Relay Servomechanisms 11. Nonlinear Systems 12. Linear System with Variable Coefficients 13. Control Design by Perturbation Theory 14. Control Design with Specified Criteria 15. Optimalizing Control 16. Filtering of Noise 17. Ultrastability and Multistability 18. Control of Error.

The treatment of these eighteen subjects is very unequal. The early ones, which may be called 'classical', have been discussed in careful detail in a number of textbooks, and their present cursory treatment is often disappointing. In some of the more advanced chapters, the author frankly states that he is reproducing a treatment already available in book form elsewhere. There is not much one can expect of a one-chapter treatment of non-linear systems! There remains the fact that about one half of the book consists of very valuable reports on advanced methods of design, recently published in aeronautical journals, some due to the author himself. It would be ungrateful to complain of presentational shortcomings; we should rather be thankful for this up-to-date information coming from a competent and very busy expert.

P. LE CORBEILLER

Studies in mathematics and mechanics. Presented to Richard von Mises by friends, colleagues, and pupils. Academic Press, Inc., New York, 1954. ix + 353 pp. $9.00.

This dedicatory volume was presented to Richard von Mises on the occasion of his seventieth birthday, April 19, 1953. Professor von Mises survived the event by only three months.

The papers were contributed from among his many students, collaborators, and friends. The breadth of subject matter is itself indicative of the manifold interests of the recipient, one of that handful of scientists who have done so much to give mechanics the shape it has today.

The papers are grouped into five categories: algebra, number theory, and geometry; analysis; theoretical mechanics; applied mechanics; probability and statistics.

The contributing authors are: E. Bompiani, Alfred Brauer, M. Herzberger, Gaston Julia, A. Ostrowski, Hans Rademacher, Francesco Severi, G. Szegö, Olga Taussky, C. Arf, Stefan Bergman and M. M. Schiffer, Garrett Birkhoff, J. B. Diaz and Alexander Weinstein, F. H. van Den Dungen, Zeev Nehari, Mauro Picone, D. C. Spencer, G. Temple, J. M. Burgess, P. R. Garabedian, G. Kuerti, C. C. Lin, Charles Loewner, G. S. S. Ludford, N. Minorsky, G. Pólya, William Prager, J. L. Synge, P. W. Bridgman, Howard W. Emmons, Karl Federhofer, Alfred M. Freudenthal, H. Reissner and E. Reissner, A. Signorini, Arthur H. Copeland, Sr., Maurice Fréchet, François N. Frenkiel and James W. Follin, Jr., Y. Garti et T. Consoli, Hilda Geiringer, Corrado Gini, J. Neyman and E. L. Scott, Nakibe T. Uzgören.

The introduction is by Professor Phillip Frank and includes a list of Professor von Mises published works. It was with delight that the reviewer noted that the list concludes with six publications on the Austrian poet, Rainer Maria Rilke. In this age of specialization they bear silent witness to the stature of this great scholar.

WILLIAM H. PELL

Automatic feedback control system synthesis. By John G. Truxal. McGraw-Hill Book Co., Inc., New York, Toronto, London, 1955. xiii + 675 pp. $12.50.

The bulk of the material in this book has been taught in a two-semester graduate servomechanisms course in the Electrical Engineering School at Purdue University. It is not, therefore, a book for the beginner; but it is, by far, the best book I have seen dealing with Automatic Control beyond the elements. Chapters 1 to 6 deal with system synthesis, mostly in the frequency domain; it includes a great deal of material relative to electric network synthesis, without which much of what is said about improving automatic control risks remaining theoretical and inapplicable. The very valuable method of signal flow diagrams, of S. J. Mason, is made the basis of feedback and stability theory. At the same time, the author lays great stress upon the necessity of correlating harmonic steady-state with transients; this requires that the student should have a real understanding of the Laplace transform, instead of the usual scanty knowledge limited to third-order transients. The method of Bubb (based on articles by Schoenberg in QAM, 1946) receives due attention. Contributions by many authors are not simply quoted, but always thoroughly explained and integrated into the general treatment. Chapers 7 to 11 describe more advanced methods of statistical design and analysis of sampled-data and non-linear systems. This very thorough, conscientious and judicious work is confidently recommended.

P. LE CORBEILLER

The elements of probability theory and some of its applications. By Harold Cramer. John Wiley & Sons, Inc., New York, and Almquist & Wiksell, Stockholm, 1955. 281 pp. $7.00.

This is a translation and revision of a book originally written in Swedish. It is designed for students at the advanced undergraduate level and should indeed be understandable to readers familiar with elementary calculus and its usual prerequisites. The emphasis is particularly well suited to those who are primarily interested in acquiring enough familiarity with probability and statistics to handle routine practical problems and should be quite valuable as a book for self study by non-mathematicians.

The book is divided into three parts. The first part labeled "Foundations" contains an interesting historical introduction after which the concept of probability is defined operationally in terms of frequency

ratios. A discussion of the elementary calculus of probability is followed by some of the standard combinatorial problems.

The subject of part II is "Random variables and probability distributions" in which only discreet or continuous distributions are considered. The discussion here is already directed heavily toward practical statistics with primary emphasis on the special types of distributions that occur frequently in statistical analysis and such concepts as regression coefficients, correlation coefficients, etc.

The third part is described as "Applications" and deals almost exclusively with problems such as sampling, statistical inference, errors and other such topics dealing with statistical analysis of finite sets of observed values of random variables.

Except possibly for a few sections in which the author attempts with only moderate success to bring some of the more advanced topics down to an elementary level, the difficulty of the book is kept at a fairly uniform and modest level with numerous examples and exercises. The level is perhaps even too elementary for someone with strong mathematical inclinations and is more suited for someone who finds mathematics difficult but necessary. There are numerous suggestions of more advanced topics that the reader may find elsewhere.

G. NEWELL

Einfuhrung in die Verbandstheorie. By Hans Hermes. Springer-Verlag, Berlin, Göttingen, Heidelberg, 1955. vi + 164 pp. $3.35.

Prof. Hermes' book makes a very pleasant introduction to some of the most interesting ideas of lattice theory, for those who are curious about this new branch of abstract algebra. It is, perhaps, even more readable than the recent comparable French book "Théorie des treillis", by Mme. Dubreil-Jacotin and MM. Lesieur and Croisot.

After introducing the basic concepts of lattice, modular and distributive lattice, and Boolean algebra, the author explains some important applications of lattice theory to other fields of mathematics, and does this very clearly. Thus, the relations to projective geometry and totally disconnected bicompact topological spaces are treated, together with applications to the structure of abstract groups. "Atomistic" Boolean algebras are treated in some detail, and the relation of lattices to logic is briefly sketched.

Throughout, the emphasis is on a clear presentation of esthetically appealing fundamental ideas; no attempt is made at completeness. The orientation is that of an algebraist and logician; not that of an analyst. Thus lattice-ordered groups, vector lattices, and partly ordered sets are not even mentioned.

Finally, it is not yet clear to what extent the ideas of lattice theory will prove fruitful in "applied mathematics", and the reader who wishes to learn about the subject with such applications in mind must be ready to develop his own techniques.

GARRETT BIRKHOFF

Mathieusche Funktionen und Sphaeroidfunktionen. By J. Meixner and F. W. Schaefke. Springer-Verlag, Berlin, Göttingen, Heidelberg, 1954. XII + 414 pp. $7.65.

Both Mathieu functions and spheroidal wave functions arise from a separation of variables of the wave equation in certain systems of curvilinear coordinates. Their importance as a tool in mathematical physics has been recognized long ago, but the development of the theory of these functions has been rather slow. Large portions of the present book by Meixner and Schaefke could not have been written fifteen years ago. It is the first account of a systematic theory of spheroidal functions, and for Mathieu functions the only comparable work is "Theory and application of Mathieu functions" (Oxford, 1947) by N. W. McLachlan.

The book under review consists of four chapters, (approximately equal in length).

In the first chapter, the authors state and prove basic theorems some of which are not directly connected with the theory of special functions. The sections on eigenvalue problems involving two parameters deserve special attention. In the statement and proof of the theorems the terminology and the basic concepts of the theory of Banach spaces are used; a brief explanation of the vocabulary is

given in small print. The results of these sections, including an equiconvergence theorem, are used extensively in the later parts of the book. In another section of Chapter I, series expansions in terms of Hankel functions of order $v + n$ (v fixed, $n = 0, \pm 1, \pm 2, \ldots$) are investigated. A useful lemma is proved stating a sufficient condition under which the asymptotic expansion of the sum of a series can be found by a term-by-term application of the asymptotic expansion for the Hankel functions.

Chapters II and III deal respectively with the theory of Mathieu functions and the theory of spheroidal wave functions. A common feature of both chapters is the underlying point of view; all functions are considered as analytic functions of both the variable and the parameters. This point of view is fully justified by the success achieved by the authors. A remarkable example is furnished by the results about the analytic behavior in the large of the characteristic curves. (In the case of Mathieu functions, these are the curves in the plane of the parameters which separate the regions of stability from those of instability.) Both Chapter II and III contain results of the following types: (1) formulas connecting the solutions defined by initial or boundary conditions for a finite interval with solutions defined by their asympotic behavior; (2) general expansion theorems; (3) addition theorems of great generality which are based on (2) and on the following remarks: The equation $\Delta u + u = 0$ is invariant under notation and translation of the coordinate system, and a solution which vanishes sufficiently strongly at infinity is identically zero.

The derivation of the results sheds a new light even on the well-known addition theorem for Bessel functions. It is not possible to describe in detail the large number of results in these chapters, nor even to point out those which are new or of special importance.

About thirty applications to problems of mathematical physics are briefly but lucidly discussed in Chapter IV. Problems in mechanics, acoustics, electromagnetics and quantum mechanics are treated. They include the diffraction and radiation problems investigated by Bouwkamp and by Stratton and Chu which started the revival of interest in spheroidal wave functions and the rapid development of the theory. Some of the applications are as recent as the problem of the strong-focusing synchrotron by E. D. Courant, M. S. Livingston and H. S. Snyder. Other applications are listed in the introduction.

The book is organized with great care and circumspection; its program and its limitations are clearly defined in the introduction, and the list of references is comprehensive and approaches completeness. The style is clear although of necessity condensed in many places. The reviewer did not feel it necessary to check the formulas for errors or misprints since this has already been done by C. J. Bouwkamp who is not only an authority on the field but also a mathematican with an unsurpassed reputation for reliability.

The work by Meixner and Schaefke can be used both as a textbook (on a reasonably high level) and as a handbook of reference for anyone who is not entirely unaquainted with the subject. It can be expected to be eminently useful and highly stimulating both in applied research and for the pure analyst.

W. Magnus

IX INTERNATIONAL CONGRESS FOR APPLIED MECHANICS

THE 9TH INTERNATIONAL CONGRESS FOR APPLIED MECHANICS will be held in Brussels in the buildings of the "Université Libre" (50, Avenue F. D. Roosevelt) **from Wednesday 5 September to Thursday 13 September 1956.**

The technical sessions of the Congress will be organized in two sections, viz.:

Section 1: Fluid dynamics and aerodynamics

Section 2: Mechanics of solids (rigid dynamics, vibrations, elasticity, plasticity)

Moreover, several general lectures are contemplated on subjects likely to be of interest to members of both sections.

It should be noted that Thermodynamics and Computational Methods as such are not included, although specific applications to pertinent problems of one of the two sections mentioned above are acceptable subjects for papers to be presented at the 9th Congress.

During the first four days of the Congress the mornings will be reserved for session of Section 1 and the afternoons for sessions of Section 2. In the second four days Section 2 will meet in the mornings and Section 1 in the afternoons. This arrangement will provide for adequate time for subsection or private meetings and also for the possibility to attend lectures in both sections.

In order to allow enough time for presentation and discussion, the Organizing Committee will make a selection. According to the time available, only a certain number of papers, chosen for their general interest, will be presented before the sections. A period of 30 minutes (20 for presentation and 10 for discussion) will be allowed for each of these papers. The remaining papers, grouped by subjects, will be presented in subsection meetings, a period of 15 minutes being allowed to each and at the end of every meeting, a general discussion will take place.

Abstracts of papers should be submitted in 3 copies to the address above **before 15 April 1956,** in order to permit these abstracts to be selected and printed before the Congress. All abstracts will be included in the book of abstracts. The book will be available upon registration at the Congress at a price of 100 F.B. (2 U. S. dollars).

The Congress fee will be 200 F.B. (4 U. S. dollars). An extra charge of 150 F.B. (3 U. S. dollars) will cover the banquet (including probably an excursion on September 10th.).

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SUGGESTIONS CONCERNING THE PREPARATION OF MANUSCRIPTS FOR THE QUARTERLY OF APPLIED MATHEMATICS

The editors will appreciate the authors' cooperation in taking note of the following directions for the preparation of manuscripts. These directions have been drawn up with a view toward eliminating unnecessary correspondence, avoiding the return of papers for changes, and reducing the charges made for "author's corrections."

Manuscripts: Papers should be submitted in original typewriting on one side only of white paper sheets and be double or triple spaced with wide margins. Marginal instructions to the printer should be written in pencil to distinguish them clearly from the body of the text.

The papers should be submitted in final form. Only typographical errors may be corrected in proofs; composition charges for all major deviations from the manuscript will be passed on to the author.

Titles: The title should be brief but express adequately the subject of the paper. The name and initials of the author should be written as he prefers; all titles and degrees or honors will be omitted. The name of the organization with which the author is associated should be given in a separate line to follow his name.

Mathematical Work: As far as possible, formulas should be typewritten; Greek letters and other symbols not available on the typewriter should be carefully inserted in ink. Manuscripts containing pencilled material other than marginal instructions to the printer will not be accepted.

The difference between capital and lower-case letters should be clearly shown; care should be taken to avoid confusion between zero (0) and the letter *O*, between the numeral one (1), the letter *l* and the prime ('), between alpha and *a*, kappa and *k*, mu and *u*, nu and *v*, eta and *n*.

The level of subscripts, exponents, subscripts to subscripts and exponents in exponents should be clearly indicated.

Dots, bars, and other markings to be set *above* letters should be strictly avoided because they require costly hand-composition; in their stead markings (such as primes or indices) which *follow* the letter should be used.

Square roots should be written with the exponent ½ rather than with the sign $\sqrt{}$.

Complicated exponents and subscripts should be avoided. Any complicated expression that recurs frequently should be represented by a special symbol.

For exponentials with lengthy or complicated exponents the symbol exp should be used, particularly if such exponentials appear in the body of the text. Thus,

$\exp[(a^2 + b^2)^{1/2}]$ is preferable to $e^{(a^2+b^2)^{1/2}}$

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